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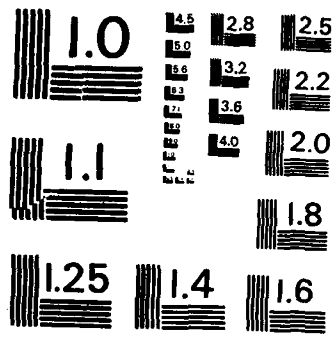

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REGULARITY FOR A SINGULAR CONSERVATION LAW

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UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

The main structure underlying the nonlinearity of conservation laws of gasdynamical type in two independent variables is discussed at the hand of a canonical example describing also properties of water waves near shore. The ultimately singular nature of such laws is here the central issue and calls for an unusual formulation. Attention is directed to the globally strong solutions, and an unusual regularization is employed to make them accessible, after illposedness is overcome. The usual regularity theory is not normally sufficient for singular partial differential equations, and the necessary additional chapter on extensions to the singular locus is developed in detail for the canonical example. Criteria for the relation between regularized and strong solutions are discussed and used to characterize the class of solutions that are globally strong in the strictest sense.

AMS (MOS) Subject Classifications: 35L65, 35L80, 35R25, 76B15

Key Words: Nonlinear Partial Differential Equations; Waves on Beaches ,

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SIGNIFICANCE AND EXPLANATION

This report discusses aspects of conservation laws and of waves on beaches close to shore.

Conservation laws of the type governing gas dynamics and explosion theory have attracted much attention in this century because they are strikingly nonlinear wave equations. This report addresses the question: What concrete structure is here hinted at by the term "nonlinear" and is responsible for the tendency of solutions to "blow up", and how can this underlying structure be crystallized and illuminated? It will be shown to be related to a wider question: We are used to finding the structure of ordinary differential equations crystallized in the singularities of the coefficients in those equations; should not something similar be expected of partial differential equations? The report makes a determined foray into this little-explored field, in order to open a new window on the connection between nonlinearity and singularity in partial differential equations.

To that end, a definite conservation law in two independent variables is here studied, which is both very simple and very typical of the gasdynamic class of such laws, in effect, a canonical example offering scope for great lucidity. It is also an oceanographical model of properties of water waves over a beach of small slope and holds the key to shore reflection, which is a sorely missing link in the application of wave theory to coastal oceanography.

Earlier work by the author and his collaborators elucidated a very singular solution class of this mode. This report concentrates on the more regular solution classes and shows how the singularity of the partial differential equations controls them also.

The report is one of a triplet, of which another explains the unusual formulation while the last, gives applications to coastal oceanography and engineering.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

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REGULARITY FOR A SINGULAR CONSERVATION LAW

R. E. Meyer

1. INTRODUCTION

The singularities of the coefficient functions in ordinary differential equations are well-known to crystallize their solution structure; for linear analytic equation, in fact, they determine it completely. Something similar would plausibly be expected of partial differential equations, but the issue has received little attention yet. It leads in unfamiliar directions, moreover, away from the mainstream of contemporary theory. ice it is a very large subject, early exploration is best focused on specific question. i gas dynamics has helped to precipitate one coherent set of problems arising from its inear conservation laws.

All of those are capable of degeneracy, and it was a surprising observation th the degeneracy of steady supersonic flow at an axis of symmetry [1], that of unsteady one-dimensional, and steady two-dimensional, motion at vacuum [2], and that of shallow-water equations at a dry-line [3], share a common mathematical representation. It was an even more intriguing observation that this degeneracy caused not only local singularity of solutions [1, 4, 5], but could have a notable, long-range influence on solution structure quite far from the singular locus of the conservation laws [4, 6, 7]. A suspicion arose that this degeneracy might crystallize a major component of the underlying structures that make those conservation laws "nonlinear" [8]. One aim of the following is to elucidate more definitely why the degeneracy does indeed crystallize the tendency to "blow-up" for conservation laws of gas dynamical type (Section 5).

These observations suggested a scope for a theory of quite a large class of conservation laws from the point of view of a common denominator. There are also very appreciable differences, however, between the differential equations of the examples just

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mentioned. The rigorous, local theory of one of them [9] shows how much the distinguishing features can add to the apparatus describing the solutions in detail. At the present stage of exploration, it may be more helpful to proceed in the opposite direction of illuminating the heart of the matter at the hand of a particularly typical example stripped of all complications that are not central to the singularity structure. Such a canonical example is offered by the equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(h + \frac{1}{2}u^2\right) = -1$$

for two functions $h(x,t)$ and $u(x,t)$ first proposed by Stoker [3] as a speculative model of waves on beaches. This is the system studied directly in the following, with only brief comments (Section 2) on the complications arising for other systems of the class of which the example illuminates the key features.

Experience with singular points of ordinary differential equations suggests that some solutions of singular conservation laws may be relatively "regular", while others may be more "singular." Since the latter are more exciting, earlier investigations concentrated upon them. One class was investigated rather thoroughly for axisymmetric supersonic flow [1, 9, 10]. A much more singular class was discovered for the canonical example [5, 7]. Preoccupation with possible "pathology", however, tends to create biased impressions obscuring the overall picture. To create a balanced basis for a singularity theory of conservation laws, the present investigation concentrates on the "more regular" solutions. It will reveal a background-hierarchy of solution regularity ordered by a concept of "n-compatibility" (Section 3). The earlier investigations are then seen to concern $n = -1$ [1, 9, 10] and $n = -2$ [5, 7], while the present one concentrates on $n > 0$. 'Strong' solutions of the conservation laws as differential equations then become important and this raises the familiar issue of their tendency to "shock-formation" or "blow-up": most, if not all, strong solutions have a very restricted domain of existence, which is awkward to predict and describe. In any case, they have ceased to exist long before they can display the singularity which this account will demonstrate to be the essential cause of their "blow-up". In fact, there is evidence [1, 11] against the existence of strong solutions that are global in any strict sense. On the other hand, a few special examples of such solutions have been found [12, 13], and the question arises whether they are isolated, special cases or whether they form a generic set of significance?

The most familiar expedient for coping with the severe restrictions on existence of strong solutions is recourse to weak solutions, but it is not helpful in the present context. Indeed, the "jump conditions" or "shock conditions" of the conservation law will not even be introduced in the following, because they lead promptly [4, 5] to the occurrence of the violent singularity [7], which then pre-empts the whole foreground so as to obscure everything else. Instead, a less familiar method of regularization [14] will be employed which associates with the conservation law a different system of differential equations.

This does not, of course, dispose of all difficulties; it shifts them to a subtler arena. It is relatively easy to imagine seeing what questions should be asked of the conservation laws. By contrast, the "apparent equations" -- to avoid burdening the text with constant repetition of the clumsily un-English word regularized -- admit a plethora of problems which look mathematically reasonable. Many have been studied, but almost all of them are academic in the sense that they generate only theorems barren of good information on the conservation laws. Many are ill-posed to some extent, but for singular partial differential equations, well-posedness in a conventional sense cannot serve as a signpost. It has taken a long time to sort the grain from the chaff, and the correct and fruitful formulation is unfamiliar in many respects. It is explained in detail, with a full motivation for it, in a related account [15] that leans heavily on physical considerations. The mathematical reasons for it would take excessive space to explain in advance because they must lean largely on hindsight from the proofs and on frustration with barren theorems. A summary (Section 2) of the formulation arrived at in [15] will therefore be more helpful here. The mathematical reasons will be mentioned where the proofs touch on them.

Additional specifications which make the "apparent problem" well-posed are introduced in Section 3 and shown to lead to existence, uniqueness, stability and regularity in the usual sense. It may be the most significant insight here gained that this is inadequate for singular partial differential equations: solution regularity becomes non-uniform near the singular locus. To understand solution structure then requires a further chapter of regularity theory which explores the existence of extensions to the singular locus. Such a theory is developed in detail in Section 4 for the canonical example.

Section 5 turns to the relation between apparent solutions and solutions of the conservation law to show why the extensions to the singular locus are also decisive for this relation. That discussion also helps to show how intimately the "nonlinear" tendency to shock-formation is linked to the singularity structure.

Those results are used in Section 6 to formulate a correct amplitude concept for a characterization of the class of solutions of the conservation law which are globally strong in the strictest sense. They do turn out to form a generic set essential for an understanding of the conservation law and, together with the singular solutions discussed by earlier studies [1, 4, 5, 7, 9, 10], offer a reasonably comprehensive, mathematical picture.

A decisive point which the picture misses is the amazing, long-range influence of the singularity upon the global structure even of regular solutions at large distances from the singular locus. That is best demonstrated at the hand of an application [16] of the present results to waves on beaches.

2. THE MODEL

The "beach equations"

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(h - h_0 + \frac{1}{2}u^2\right) = 0 \tag{2}$$

give [15 Section IV, 17] an approximate description of the local water depth $h(x,t)$ and (vertically averaged) horizontal velocity $u(x,t)$ of inviscid, irrotational motion under gravity of water with a free surface over a beach <u>of small slope</u> at time t and distance $-x$ from the undisturbed shore position (Fig. 1); $h_0(x)$ is the known, undisturbed water depth. All the variables have been made non-dimensional by reference to an unknown scaling [15 Section IV], and this places a serious, if vague, restriction on the "confidence domain" in which (1), (2) are a model of water waves. These equations must therefore be restricted to a "beach domain" $\{x,t : 0 < h(x,t) < \text{const.}\}$, and for simplicity of notation, this constant will be taken here as 1/16.

The equations are a quasi-linear, hyperbolic system with degeneracy at the 'free boundary' $h(x,t) = 0$, which represents the moving shoreline. Effective analysis of such a system must cope with its well-known tendency to "shock formation" by an extension of the solution concept, and the beach equations lend themselves to a particularly lucid exposition because an extension suitable for present purposes can be achieved trivially simply, if attention be henceforth restricted to beaches of uniform slope, so that

$$h_0(x) = -x \ .$$

Then

$$\alpha = 2h^{1/2} + u + t - \tfrac{1}{2}, \qquad \beta = 2h^{1/2} - u - t + \tfrac{1}{2} \tag{3}$$

are Riemann invariants of (1), (2), and their adoption as independent variables yields the regularization to be used here. (For other conservation laws, a less obvious choice of characteristic variables may be needed for the same purpose [14].) This step suffices, moreover, to transform the unknown, moving boundary into the fixed line

$$4h^{1/2} = \alpha + \beta = 0 \ .$$

(That the singular locus can be made explicit is typical of hyperbolic conservation laws, even if the coordinates achieving it are not normally as straightforward.)

The transformation to the independent variables (3) takes (1), (2) into

$$\frac{\partial x}{\partial \beta} = (u + h^{1/2}) \frac{\partial t}{\partial \beta}, \qquad \frac{\partial x}{\partial \alpha} = (u - h^{1/2}) \frac{\partial t}{\partial \alpha} \tag{4}$$

for $x(\alpha,\beta)$. $t(\alpha,\beta)$ on the beach domain $0 < \alpha + \beta < 1$. If this system has a single-valued solution, then

$$a(\alpha,\beta) = \frac{1}{2} (\alpha + \beta)^{3/2}\left(\partial t/\partial \alpha - \frac{1}{2}\right) , \tag{5}$$

$$b(x,\beta) = \frac{1}{2} (\alpha + \beta)^{3/2}\left(\partial t/\partial \beta + \frac{1}{2}\right) \tag{6}$$

must satisfy

$$\frac{\partial a}{\partial \beta} = - \frac{3/2}{\alpha + \beta} b , \tag{7}$$

$$\frac{\partial b}{\partial \alpha} = - \frac{3/2}{\alpha + \beta} a , \tag{8}$$

which are the "canonical equations" [5, 8] of (1), (2). Their interpretation is explained in [15 Section V,]: a and b are characteristic measures of fluid acceleration of the waves incident on, and reflected from, the beach, respectively, and (7), (8) give a lucid description of the mechanism of mutual generation and interaction of these waves. The coefficient $(\alpha + \beta)^{-1} = h^{-1/2}/4$ crystallizes the singular enhancement of this mechanism as the local water depth $h(x,t) \downarrow 0$; it is the key to all that follows.

(That the canonical equations are linear, is a property of the canonical example which helps greatly to make the analysis more lucid. A first impression of the complications arising with other conservation laws may be gained from brief thought about (1), (2) with $h_0'(x) \neq$ const. All the explicitness of (3)-(8) is then lost, but the essence remains quite unchanged [18], as long as $h_0(x)$ is reasonably smooth and $h_0'(0) < 0$. For more general, hyperbolic conservation laws in two independent variables, the canonical equations are a system of four nonlinear, first-order, partial differential equations. That adds much to the labor; the contractions then necessary have been explored in [9]. It is there seen clearly, however, that the crux is always the nature of the singularity of the coefficients in the canonical equations, and when the implications of this singularity are analyzed correctly, the additional contractions fall into place [9]. Of course, fully global descriptions are not normally achieved thereby, for instance, the influence of the axis extends quite far in axisymmetric, steady, supersonic flow [10], but not to very large radii, and the analysis [1, 9] focusing upon the singularity on the axis does not begin to touch the transonic singularity. The effective domain of such analysis is therefore generally subject to limitations somewhat analogous to the restriction of the beach equations to their "confidence domain". For these reasons, (1), (2) with $h_0(x) = -x$ typify the key features, stripped of all obscuring complications, of hyperbolic conservation laws in two independent variables for which the dominant singularity of the canonical equations is a 'simple pole', as in (7), (8).)

The alternate independent variables

$$\begin{aligned} \sigma &= \alpha + \beta = 4h^{1/2}, & 2\alpha &= \sigma + \lambda \\ \lambda &= \alpha - \beta = 2t + 2u - 1, & 2\beta &= \sigma - \lambda \end{aligned} \tag{9}$$

are useful in the description of singularity structure; λ is [15 Section IV] the characteristic time and σ, the characteristic (measure of) shore distance of the beach equations. To avoid confusion, capital letters will be used to denote the dependent variables as functions of σ and λ, e.g.,

$$t(\alpha,\beta) = T(\sigma,\lambda), \qquad a(\alpha,\beta) = A(\sigma,\lambda) ,$$

etc. The singularity structure of the canonical equations prompts use of dependent variables

$$\begin{aligned} Y(\sigma,\lambda) &= a - b & Z &= a + b \\ &= \sigma^{3/2}(\partial T/\partial\lambda - \tfrac{1}{2}), & &= \sigma^{3/2}\partial T/\partial\sigma , \end{aligned} \tag{10}$$

for which (7), (8) take the form

$$\sigma^{3/2}\frac{\partial Y}{\partial\lambda} = \frac{\partial}{\partial\sigma}(\sigma^{3/2}Z) , \tag{11}$$

$$\sigma^{-3/2} \frac{\partial Z}{\partial \lambda} = \frac{\partial}{\partial \sigma} (\sigma^{-3/2} Y) \; ; \tag{12}$$

$\sigma^{3/2}Z$ and $\sigma^{1/2}Y$ are [15 Section VII] shoreward and longshore mass-flow rates, respectively. If a and b satisfy (7), (8), it follows that Y and Z must satisfy the Euler-Poisson-Darboux equations

$$\frac{\partial^2 Y}{\partial \sigma^2} - \frac{\partial^2 Y}{\partial \lambda^2} = \frac{3}{4} \sigma^{-2} Y \; , \tag{13}$$

$$\frac{\partial^2 Z}{\partial \sigma^2} - \frac{\partial^2 Y}{\partial \lambda^2} = \frac{15}{4} \sigma^{-2} Z \; , \tag{14}$$

which are linear, but singular, wave equations. For convenience, any of the regularized equations (4), (7), (8), (11)-(14) will be called indiscriminately "apparent equations", without implication of equivalence.

To complement such equations to an "apparent problem" requires specification of initial and boundary conditions. Since the original definition, $h(x,t) = 0$, of the moving shoreline has been absorbed into the notation, a need for a further condition there for each of (13) and (14) is plausible. That for (14) was found by Taylor [13]: the moving shoreline is, by definition, the line *across which there can be no* fluid mass-flow and hence, the mass-flow rate $\sigma^{3/2}Z$ must vanish there. Similarly [15 Section VII], the longshore mass-flow rate $\sigma^{1/2}Y$ must vanish there, if the velocity is to remain finite, because the water tends to zero. Hence,

$$\lim_{\sigma \to 0} [\sigma^{1/2} Y(\sigma,\lambda)] = 0 \; , \tag{15}$$

$$\lim_{\sigma \to 0} [\sigma^{3/2} Z(\sigma,\lambda)] = 0 \; , \tag{16}$$

are necessary on 'physical' grounds. Observe that existence of limits of Y and Z themselves is not implied thereby. The mathematical reasons why specification of $\lim(\sigma^{3/2}Z)$ and $\lim(\sigma^{1/2}Y)$ is appropriate, but Z or Y do not normally exist at the singular locus $\sigma = 0$, were found by Taylor [19] and Shen [20], respectively. are briefly explained in [21], and will become very clear in the existence proof in Appendix A below: they are connected intimately with the form of the singularity in (13) and (14).

Since $t(\alpha,\beta) = 0$ is a non-characteristic line for the wave equations (13), (14), the literature would suggest arbitrary Cauchy data specifying there Y, $\partial Y/\partial\lambda$ and Z, $\partial Z/\partial\lambda$. That is not at all possible for (13), (14), however, because the initial line intersects their singular locus. On the other hand, Cauchy data near a singular line of a differential equation are somewhat academic, in any case, and it will unburden the presentation considerably to defer the matter to Appendix A. Meanwhile, the significant issues can be clarified, and the stripping of inessentials from the canonical example completed, by the adoption of the undisturbed initial state.

For σ bounded from zero, (13) and (14) are regular wave equations and their classical uniqueness theorem [22] shows the undisturbed state to persist for $\alpha < 0$, $\beta > 1/2$ (i.e., in the triangle **AGC** in Fig. 2) and, by continuous extension, also $\beta > 1/2$. That defines (trivial) characteristic data on **AG** which, together with (15), (16), pose a singular problem for (13) and (14) in the triangle **AOG** (Fig. 2). It is a special case of the problem discussed in Section 3, but too simple a case for its separate analysis to generate worthwhile insight. It may suffice, therefore, to remark that the line of argument employed in the existence proof (Appendix A) demonstrates easily that the undisturbed state persists throughout **AOC** (Fig. 2). In effect, a wave incident from the sea enters the beach domain $0 < \sigma < 1$, $\lambda > -1$ at the time $t = 0$; since its front propagates along the characteristic $\alpha = 0$, by (4), it leaves the motion undisturbed for $\alpha < 0$.

The initial condition of undisturbed water at rest can therefore be stated more conveniently as

$$u \equiv h + x \equiv Y \equiv Z \equiv 0 \quad \text{for} \quad \alpha < 0, \quad \sigma > 0 \,. \tag{17}$$

While this simple initial condition adds greatly to lucidity, and also suffices to guarantee that the ensuing inviscid fluid motion is irrotational, it exacts a price: there is no way of estimating a priori how long it may take for a motion of real interest to develop. A useful theory must therefore be free of any restriction whatever on the time-interval it can cover, it must be uniform in t on $[0,\infty)$.

Since (13) to (17) are all homogeneous, the problem is clearly incomplete. The wave incident from the sea, which causes the water motion, must be specified at the outer boundary, $\sigma = 1$, of the beach domain. On mathematical grounds, the most natural choice for (13) and (14) might seem to specify Y and Z there, respectively [13, 19, 20]. However, Z is symmetrical in the incident and reflected acceleration measures, by (10), so that $Z(1,\lambda)$ describes, not the incident wave, but an equal amount of partial information on both the reflected and incident waves, and similarly for $Y(1,\lambda)$. Their specification would therefore thwart the most important objective of the theory, to analyze the process of wave reflection up to the shoreline. The physical interpretation [15 Section V] of the canonical equations indicates that a proper characterization of the incident wave crossing the outer boundary B of the beach domain (Fig. 3) is to specify there the incident acceleration measure a. By (10), however, that couples the problems for (13) and (14) in a mathematically awkward manner. Moreover, it does not amount to a direct characterization of the wave incident from the sea, which has already interacted with the reflected wave before it arrives at the sea boundary B (Fig. 3). Most of that early interaction must be expected to have occurred beyond the confidence domain and hence, cannot be analyzed within the framework of a local model. (This difficulty is not a general affliction of hyperbolic conservation laws [10].) The issue is therefore not aggravated by a modification of the incident-wave specification, as long as it does not prejudice the interaction process in the beach domain. That oblique strip-domain (Fig. 3) is itself awkward in relation to the wave equations (13), (14), of which the natural domain is the characteristic rectangle.

A gain in mathematical simplicity and lucidity can therefore be obtained from an extension of the problem [15 Section IX]: specify the incident acceleration measure a along the characteristic $\beta = 1$ (Fig. 3) as a function of α,

$$a(\alpha, 1) = \hat{a}(\alpha) \quad \text{on} \quad [0,\bar{\alpha}] \tag{18}$$

and at the same time, <u>require the apparent equations</u> (4)-(14) <u>to be extended beyond the beach domain</u> throughout the region between $\sigma = 1$ and $\beta = 1$ (Fig. 3).

From (4), the characteristic $\beta = 1$ is seen to be the boundary I of the domain of influence of apparent solutions in the beach domain and hence, (18) cannot prejudice the process of wave reflection in that domain. Since $u = h + x = 0$ initially, moreover, (18) suffices to specify the incident wave on $\beta = 1$. Admittedly, the extension of the apparent equations is somewhat abstract, because the characteristic $\beta = 1$ cannot, as α increases, remain in the confidence domain of (1), (2) as a model of water motion. A part of the extended solution cannot therefore be expected to describe properties of water waves, but that does not diminish its relevance in the confidence domain, nor the gain in mathematical clarity.

To sum up, the apparent problem is to solve (13), subject to (15), (17), (18), or (14), subject to (16), (17), (18), on the "apparent domain"

$$\sigma > 0, \quad 0 < \alpha < \bar{\alpha}, \quad \beta < 1 \tag{19}$$

bounded by (Fig. 3) the singular line Σ, the "initial" line OC, the "incidence boundary" I and the characteristic EW. The last boundary arises because (19) is a maximal domain: the incidence data (18) cannot determine the solution beyond $\alpha = \bar{\alpha}$. Time-uniformity of the theory is equivalent to absence of any restriction on the choice of $\bar{\alpha}$ in (18).

It will be observed that, if the beach equations (1), (2) have a strong solution in a domain, then the characteristic transformation maps it on a definite, apparent solution on the image of that domain. By contrast, if the apparent problem has a solution, it will exist on the maximal domain (19), by the linearity of (13), (14). That is the sense in which the apparent problem regularizes the real, beach problem.

One may wish that the relation between beach solution and apparent solution be one-one, when it exists, and that can be assured by a minor qualification: for the beach equations, (18) specifies the incident wave on the characteristic line $x = x_I(t)$ of which the incidence boundary I is the image. There $\beta(t) = 1$, so $\alpha = \alpha_I(t) = 2(u + t)$, which is a strictly monotone function only if

$$du/dt > -1 \text{ along } I\ . \tag{20}$$

That inequality is therefore a hidden restriction on (18). If it be unacceptable, however, it suffices to replace α by a suitable function $\chi(\alpha)$ as independent variable; in (18), $\hat{s}(\alpha)$ is replaced by a function $\hat{s}(\chi)$, but nothing of substance is changed. In the interest of maximal lucidity of notation, acceptance of (20) will be preferred here.

3. EXISTENCE

The apparent problem is still ill-posed, however. A preliminary step in coping with that concerns the function class of the data (18). If the apparent equations were regular, any class could serve equally well because the solution would preserve the regularity of the data [22]. For the singular EPD equation, however, this fails: in terms of continuity classes, e.g., if the input function $\hat{a}(\alpha) \in C^1[0,\bar{\alpha}]$ in (18), then Y and Z may be only in C^0 in the interior of the domain, not to mention their non-existence at the singular line. A compromise that can preserve symmetry between input and output was found by Taylor [19]:

Definition. $F(\sigma,\lambda) \in S_L[K]$ on a subset K of the apparent domain (19) means that F satisfies a Hoelder condition,

$$|F(\sigma',\lambda') - F(\sigma,\lambda)| < M(K)(|\sigma' - \sigma|^\gamma + |\lambda' - \lambda|^\gamma) \tag{21}$$

for every $\gamma < 1/2$ and for all (σ,λ) and (σ',λ') in K. $F(\sigma,\lambda) \in S_L^n$ means that all its partial derivatives of order $\leq n$ are in $S_L(=S_L^0)$.

To make the apparent problems well-posed requires, first, enough smoothness and an adjustment of the data (18) to the initial condition (17), and secondly, that any more direct input functions for (13) or (14) on the incidence boundary I (Fig. 3) be generated from (18) with careful regard to consistence with the canonical equations (7), (8).

Definition. n-compatible incidence data means both a function $\hat{a}(\alpha) \in S_L^n[0,\bar{\alpha}]$ in (18) with derivatives $\hat{a}^{(k)}(0) = 0$ for $0 \leq k \leq n$, and also use of (7), (8), (10) [together with the partial derivatives of order $\leq n$ of those equations] and of $a(\alpha,1) = \hat{a}(\alpha)$ to define functions $b(\alpha,\beta)$, $y(\alpha,\beta)$, $z(\alpha,\beta)$ and their partial derivatives of order $\leq n$ on the incidence boundary $\beta = 1$ so that $\partial^k b/\partial\beta^k = 0$ at $\alpha = 0$ for $0 \leq k \leq n$.

It is here understood that only non-negative integers n will be considered in the following, and it will be observed that (n + 1)-compatibility implies n-compatibility: the compatibility classes form a nested sequence.

Incidence Theorem. Given 0-compatible incidence data, there exist unique solutions $Y(\sigma,\lambda)$ of (13) and $Z(\sigma,\lambda)$ of (14) on the apparent domain (19) which satisfy (15) to

(18). These apparent solutions, moreover, depend continuously on $\hat{a}(\alpha)$ and are in $S_L[K]$ on every compact subset K of the apparent domain disjoint from the singular line $\sigma = 0$.

Corollary. For n-compatible incidence data, the apparent solutions Y and Z are in $S_L^n[K]$.

Incidence Corollary. For n-compatible incidence data, the theorem applies to $\partial^n Y/\partial\lambda^n$ and $\partial^n Z/\partial\lambda^n$ in the place of Y and Z, respectively.

A proof of the theorem and corollaries is given in Appendix A along lines developed by Taylor [19]. It should be observed that the proof leaves the number $\tilde{\alpha}$ arbitrary, so that the theorem and corollaries respect the requirement of time-uniformity. The following lemma is also proved in Appendix A.

Corollary (Shen's lemma [21]). If the incidence data are 1-compatible, then $Y + Z = 2a$ and $Z - Y = 2b$ satisfy the canonical equations (7), (8) on the apparent domain.

The disastrous feature of these theorems, however, is the uniqueness: more than the mass-flow conditions (15), (16) cannot be imposed at the singular line without jeopardy to the existence. But by (10),

$$\sigma^{1/2}Y = \sigma^2\left(\partial T/\partial\lambda - \tfrac{1}{2}\right), \qquad \sigma^{3/2}Z = \sigma^3\partial T/\partial\sigma ,$$

so that neither (15) nor (16) implies existence of time T or position X on the singular locus, and it is meaningless to talk of mass-flow rates where time and position cannot be defined!

Such qualms are not far-fetched, moreover. The singular solution class studied in [7], and shown in [23] to possess striking physical realism, has the property that all positive clock-times T occur at a single point of the singular locus and neither T nor X exist on it at any later characteristic time λ. The mass-flow conditions (15), (16), while physically necessary, are irrelevant to the analysis [4, 5, 7] of that solution class.

This suggests that it might have been better to study, instead of (13) or (14), the corresponding wave equation for $t(\alpha,\beta) = T(\sigma,\lambda)$ itself. The analogous theorem for the apparent problem for T, however, shows it to be well-posed when no more than

$$\lim_{\sigma \to 0} [\sigma^2 T(\sigma,\lambda)]$$

is prescribed. An extension of the clock-time T to the singular locus does not normally exist, and that destroys the chance for a useful relation between the apparent problem and the real problem for (1), (2): a globally well-posed, apparent solution need not correspond to a globally meaningful solution of the conservation law.

Definition. An apparent solution is admissible means

$$\lim_{\sigma \downarrow 0} T(\sigma,\lambda) \;\exists \quad \text{for each fixed} \quad \lambda \in [0,2\bar{\alpha}] \; . \tag{22}$$

While this postulates only a conditional extension of time to the singular locus, by contrast to a proper extension by continuity, it implies [15, Section X] both the mass-flow conditions (15) and (16). The converse is demonstrated to be false by the theorems just discussed, however much it be needed for the analysis.

4. REGULARITY

These paradoxa pinpoint an issue of central significance for singular partial differential equations. Normal regularity theory concerns the nature of solutions in the interior of the domain and its results are non-uniform for singular differential equations. A further chapter is needed to discuss conditions on the data which can assure extensions of relevant solution properties to the singular locus. For the problem class here studied, it can be based on an estimate demonstrating more regularity for all apparent solutions than the incidence theorem had revealed:

Regularity Theorem. The apparent solutions for n-compatible incidence data satisfy

$$\lim_{\sigma \to 0} (\sigma^{\varepsilon}\, \partial^n Y/\partial\lambda^n) = \lim_{\sigma \to 0} (\sigma^{1+\varepsilon}\, \partial^n Z/\partial\lambda^n) = 0 \quad \forall\, \varepsilon > 0$$

and for $0 \leq \lambda \leq 2\bar{\alpha}$.

Even though it may seem curiously indefinite, this theorem is, in fact, essentially sharp: the problem admits [1, 24] logarithmic singularities in the domain, even if none are apparent in the incidence data. A proof of the theorem is given in Appendix B; its ε, of course, is the $1/2 - \gamma$ of the S_L-definition (Section 3). As usually, the significance of the theorem lies in the corollary sequences which it generates, and of which proofs will be found in Appendix C.

Corollary R1. If the incidence data are $(n + 1)$-compatible for $n \geq 0$, then for any $\varepsilon \in (0,5/2)$,

$$\sigma^{\varepsilon-1}\partial^n Z/\partial\lambda^n \to 0 \quad \text{as} \quad \sigma \to 0 \quad \text{for fixed} \quad \lambda \,.$$

Admissibility Theorem. 1-compatibility of the incidence data on $[0,\bar{\alpha}]$ assures admissibility of the apparent solution for $0 \leq \lambda < 2\bar{\alpha}$.

Corollary R1a. 1-compatibility of the incidence data assures the assumptions of Shen's Boundedness Theorem [20, 21].

That theorem formulates conditions which imply non-existence of a resonance mechanism in the water-wave model represented by the beach equations (1), (2).

Corollary R2. For 2-compatible incidence data on $[0,\bar{\alpha}]$, $\sigma^{-3/2}Y = \partial T/\partial\lambda - \frac{1}{2}$ tends to a limit as $\sigma \to 0$ for each fixed $\lambda \in [0,2\bar{\alpha})$.

Admissibility Corollary. For 2-compatible incidence data on $[0,\bar{\alpha}]$, clock-time $T(\sigma,\lambda)$ has an extension by continuity to the segment $0 < \lambda < 2\bar{\alpha}$ of the singular line $\sigma = 0$.

Corollary R3. For 3-compatible incidence data on $[0,\bar{\alpha}]$, $\sigma^{-5/2}Z(\sigma,\lambda) = \sigma^{-1}\partial T/\partial\sigma$ tends to a limit as $\sigma \to 0$ for any fixed $\lambda \in [0,2\bar{\alpha})$ (and therefore, $\partial T/\partial\sigma \to 0$ as $\sigma \to 0$).

Corollary R3a. For 3-compatible incidence data on $[0,\bar{\alpha}]$, $\sigma^{-3/2}Y = \partial T/\partial\lambda$ has a continuous extension to the singular line $\sigma = 0$ for $0 < \lambda < 2\bar{\alpha}$.

Corollary R4. For 4-compatible incidence data, $\sigma^{-5/2}Z$ has a continuous extension to the singular line.

The pattern of extensions obtained with increasing degree of compatibility will now be plain enough not to require spelling out in detail. It illustrates clearly that the compatibility degree classifies the regularity classes of apparent solutions. To illustrate the context of this result, it may help to recall a familiar experience with linear, analytic, ordinary differential equations: the number of distinct types of solution-singularity associated with a singular point of the equation equals the order of the differential equation. For the nonlinear conservation laws (1), (2), the corresponding number turns out, luckily, to be no more than countably infinite; the compatibility definition (Section 3) identifies the counting parameter.

It is worth observing also that the regularity proofs in Appendices B, C place no restrictions on the number $\bar{\alpha}$. For incidence data of appropriate compatibility specified on sufficiently long intervals $[0,\bar{\alpha}]$, the regularity results cover arbitrarily long intervals of characteristic time λ. Once 1-compatibility is assured, moreover, Shen's Boundedness Theorem [20, 21] shows those intervals to correspond also to arbitrarily long intervals of clock-time.

It must be stressed, however, that all the results given so far concern only the regularized, apparent solutions and that any relation to solutions of the nonlinear conservation laws (1), (2) still remains to be explored. The objective of the next section is to explain why the regularity theory of extensions to the singular line is decisive also in that respect.

5. INVERTIBILITY

To examine the meaning, if any, of an apparent solution for the beach equations (1), (2), it is necessary to invert the characteristic transformation of Section 2. The familiar condition that the Jacobian $J = \partial(x,t)/\partial(\alpha,\beta)$ do not vanish is known to be insufficient because it is only a local condition [e.g., 25 and references there cited]. For degenerate characteristic transformations, it may also be unnecessary. In the present context, the following lemma is shown in Appendix D to give a sufficient condition.

Invertibility Lemma. If an apparent solution for 1-compatible incidence data has both the properties

(i) $\partial t/\partial\alpha > 0$ and $\partial t/\partial\beta < 0$ for $\sigma > 0$,

(ii) $2\partial T/\partial\lambda = \partial t/\partial\alpha - \partial t/\partial\beta > \delta_1$ for $\sigma > 0$ and some $\delta_1 > 0$,

then the characteristic transformation is invertible on the closure of the apparent domain.

At first sight, the use of the lemma will appear to hinge on the initial signs of $\partial t/\partial\alpha$ and $\partial t/\partial\beta$ and hence, on the choice (17) which makes $\partial t/\partial\alpha = -\partial t/\partial\beta = 1/2$ initially, by (5) and (6). However, the discussion of initial data in Appendix A shows them to be compatible with the shore conditions (15), (16) only if those particular values of $\partial t/\partial\alpha$ and $\partial t/\partial\beta$ are approached on the initial line as $\sigma \to 0$.

The properties (i), (ii) in the lemma are not of the kind that can be established by regularity theory alone, but much light can be shed on the way in which the issue must be resolved by combining that theory with a fundamental feature of hyperbolic conservation laws based on their canonical equations [8]:

Monotoneity Theorem. (A) Let a characteristic rectangle be such that $\alpha + \beta = \sigma > \sigma_1 > 0$ on it and assume that $\partial t/\partial\alpha > 0$ and $\partial t/\partial\beta < 0$ on the closures of the rectangle sides on which α and β take their respective minima over the rectangle closure. Then, for 1-compatible incidence data, the apparent solution has the properties (i) and (ii) of the Invertibility Lemma on the closure of the rectangle.

(B) Again, let a triangle in the characteristic plane be such that two sides are characteristic (Fig. 4) and $\sigma = \text{const.} = \sigma_1 > 0$ on the third side and $\sigma > \sigma_1$ in the triangle, and assume $\partial t/\partial\alpha > 0$, $\partial t/\partial\beta < 0$ on the closure of the side on which $\sigma = \sigma_1$.

Then for 1-compatible incidence data, the apparent solution has the properties (i) and (ii) on the closure of the triangle.

Proof. The functions

$$a^+(\alpha,\beta) = a(\alpha,\beta) + \sigma^{3/2}/4 = \frac{1}{2}\,\sigma^{3/2}\partial t/\partial\alpha\ ,$$

$$b^+(\alpha,\beta) = b(\alpha,\beta) - \sigma^{3/2}/4 = \frac{1}{2}\,\sigma^{3/2}\partial t/\partial\beta\ ,$$

are renormalizations of the characteristic acceleration measures in (5), (6) which leave the canonical equations (7), (8) unchanged:

$$\partial a^+/\partial\beta = -\frac{3}{2}\,(\alpha+\beta)^{-1}b^+\ , \tag{23}$$

$$\partial b^+/\partial\alpha = -\frac{3}{2}\,(\alpha+\beta)^{-1}a^+\ . \tag{24}$$

Moreover,

$$a^+ - b^+ = \sigma^{3/2}\partial T/\partial\lambda\ ,$$

by (9). In this notation, the conditions (i), (ii) of the Invertibility Lemma read

$$a^+ > 0,\ \ b^+ < 0 \qquad \text{for}\ \ \sigma > 0 \tag{i}$$

$$\sigma^{-3/2}(a^+ - b^+) > \delta_1 > 0\ \ \text{for}\ \ \sigma > 0 \tag{ii}$$

Consider first part (B) of the theorem and recall from the Incidence Theorem and Shen's Lemma (Section 3) that 1-compatibility assures continuity of a^+ and b^+ for $\sigma > 0$ and validity of the canonical equations. As long as $b^+ < 0$, therefore, a^+ increases with β at fixed α, by (23), from the positive values it takes, by hypothesis, on the side U of the triangle on which $\sigma = \text{const.} = \sigma_1$ (Fig. 4). Hence, if a root of a^+ be found in the triangle (Fig. 4), then a root of b^+ must also be found at the same α and smaller β. But similarly, by (24), if a root of b^+ be found in the triangle, then one of a^+ must also be found at the same β and smaller α. Either root therefore implies a succession of roots of a^+ approaching the triangle side U arbitrarily closely (Fig. 4). That contradicts the fact that $a^+ > 0$ near U, by continuity.

Hence, $a^+ > 0$ and $b^+ < 0$ throughout the closure of the triangle and by (23), (24), a^+ increases with β at fixed α, while b^+ decreases with increasing α at fixed β. It follows that

$$a^+ - b^+ > \min a^+ - \max b^+ > 0 \tag{25}$$

on the triangle closure, where the extrema are those over the closure of the triangle side U (Fig. 4).

The proof of (A) is quite analogous and leads to the same inequalities, except that the extrema in (25) are then over the rectangle sides on which α and β, respectively, take their minima over the rectangle.

The monotoneity argument of the proof illuminates the key role of the canonical equations in the structure of the beach equations [5] and similar conservation laws [8]: the canonical equations tell the directions in which characteristic accelerations are amplified by the 'nonlinear' process of wave generation and interaction.

Invertibility Corollary. If an apparent solution for 1-compatible incidence data possesses the property (i) of the lemma for $0 < \lambda < 2\alpha_1$ on a line $\sigma = \text{const.} = \sigma_1 > 0$ (Figs. 4, 5), then the apparent solution is invertible for all $\sigma > \sigma_1$ and $\alpha < \alpha_1$ in the apparent domain.

Proof. Apply part (B) of the Monotoneity Theorem to a triangle I (Fig. 5) with the same σ_1 and with $\min \alpha = 0$, and then apply part (A) to the rectangle II (Fig. 5); the Invertibility Lemma completes the proof.

The lesson is that, if we can establish invertibility along some line $\sigma = \sigma_1$, then the corollary guarantees invertibility at all greater distances from the singular line, but for $0 < \sigma < \sigma_1$, the threat to invertibility remains. As a result, local invertibility on the incidence boundary I (Figs. 3, 5) is necessary for a direct relation between apparent and proper solutions of the conservation laws (1), (2), but is not of much help in settling the issue. It can be resolved globally in a favorable sense only by its resolution in the immediate neighborhood of the singular line.

In fact, the proof of the Monotoneity Theorem demonstrates how it is the sign of the coefficient in the canonical equations (7), (8) which makes the threat to inversion decrease with increasing σ, and conversely, increase as the characteristic distance σ from the shoreline decreases. That exemplifies a more general property of hyperbo conservation laws of gas dynamical type [8]. The canonical example reveals in a particularly lucid way how the growth in magnitude of the coefficient of (7), (8) enhances this tendency as $\sigma = \alpha + \beta$ decreases, and enhances it critically as the singular line is approached. The singularity of the conservation laws thus crystallizes the global tendency to 'blow-up', which is often thought of as the characteristically 'nonlinear' feature of such conservation laws.

Ultimately, therefore, the test of global invertibility must concern extensions to the singular line, and that can be phrased strikingly for data of greater compatibility:

Inversion Criterion. An apparent solution for 3-compatible incidence data is invertible on the whole apparent domain if, and only if, it possesses the property (ii) on the singular line itself, i.e.,

$$\lim_{\sigma \to 0} \partial T/\partial\lambda > 0 \quad \text{for} \quad 0 < \lambda < 2\bar{\alpha} \,. \tag{26}$$

Proof. For reference to the regularity theory of Section 4, it is convenient to write the conditions (i), (ii) of the Invertibility Lemma in terms of Y and Z. By (10), (ii) translates into

$$\frac{1}{2} + \sigma^{-3/2} Y > \delta_1 \quad \text{for} \quad \sigma > 0 \quad \text{and some} \quad \delta_1 > 0 \,, \tag{ii}$$

and by (5), (6) and (10), (i) reads

$$\sigma^{-3/2}|Z| < \sigma^{-3/2} Y + \frac{1}{2} \quad \text{for} \quad \sigma > 0 \,. \tag{i}$$

Since the corollary already assures inversion for $\sigma > \sigma_1$, if (i) holds for $\sigma = \sigma_1 > 0$ and $0 < \lambda < 2\bar{\alpha} - \sigma_1$, it suffices now to establish (i) and (ii) for $0 < \sigma < \sigma_1$, $0 < \alpha < \bar{\alpha}$ with arbitrarily small σ_1 and then to apply the Invertibility Lemma.

By the Incidence Theorem (Section 3) and Corollary R3 (Section 4), $|\sigma^{-3/2} Z|$ is continuous for $\sigma > 0$ and arbitrarily small for sufficiently small $\sigma > 0$, and hence, (ii) will imply (i) automatically, if σ_1 be chosen small enough. In turn, (ii) then follows for $0 < \sigma < \sigma_1$ from (26) because $\sigma^{-3/2} Y$ is continuous for $\sigma > 0$, by Corollary R3a.

Conversely, since

$$\partial X/\partial\lambda = u\partial T/\partial\lambda - \frac{1}{4}\sigma\partial T/\partial\sigma \,,$$

by (4) and (9), and since $\sigma\, \partial T/\partial\sigma \to 0$ as $\sigma \to 0$ for admissible solutions (Section 3), a root of $\partial T/\partial\lambda$ at $\sigma = 0$ coincides with a root of $\partial X/\partial\lambda$ and inversion fails there, at least marginally.

6. AMPLITUDE

A useful characterization of the class of globally strong solutions of the conservation law (1), (2) requires reference to a more quantitative concept than discussed so far, which can represent an "amplitude" of the data. Invertibility of the characteristic transformation on the incidence boundary I itself (Fig. 3) is obviously necessary, and $4\hat{a}(\alpha) > -(1+\alpha)^{3/2}$ is necessary and sufficient for that, by the compatibility definition (Section 3) and the Invertibility Lemma (Section 5). A degree of compatibility is clearly necessary, as well, and the discussion of the preceding section indicates that less than 3-compatibility may be insufficient for a simple characterization of globally strong solutions. The simplest amplitude concept for them is therefore the following.

Definition: For 3-compatible incidence data $\hat{a}(\alpha) > -(1+\alpha)^{3/2}/4$ on $[0,\bar{\alpha}]$, amplitude means

$$\max_{[0,\bar{\alpha}]} |\hat{a}(\alpha)| = \delta \ .$$

Inversion Theorem. For an incident wave of sufficiently small amplitude δ, the apparent solution is invertible globally in the strictest sense.

The phrase "in the strictest sense" is here meant to convey that inversion extends over characteristic-time intervals of quite arbitrary length, provided only that the incident wave be specified over correspondingly long intervals, so that the solution remains determinate. It is worth emphasis that the phrase "sufficiently small" reflects merely the wish to present a general result susceptible of a very simple proof:

The apparent problem (13)-(18) is linear with inhomogeneous input derived only from the incidence data (18). By the compatibility definition, Y scales in proportion to δ on the incidence boundary, and hence, Y scales in proportion to δ throughout the apparent domain. By corollary R2 (Section 4) and (10), therefore, (26) follows for sufficiently small δ, and the theorem is an immediate corollary of the Inversion Criterion (Section 5).

The weakness of this result lies clearly in the woolliness of the terms "sufficiently small" and "too large". It should be observed that the theory is devoid of parameters other than $\bar{\alpha}$, on which the amplitude bound can depend only if the data be specified so that $\max|\hat{a}|$ occurs at $\alpha = \bar{\alpha}$. Since the theory is uniform in time, small or large cannot refer to coordinate bounds either. There is no rational basis, therefore, to which those terms can be linked, other than the naiveté of the amplitude definition.

There is no difficulty in defining a "sharp" amplitude for 3-compatible data. Consider the functional of the data which is

$$\min_{[0,2\bar{\alpha}]} \lim_{\sigma \to 0} \partial T/\partial \lambda \ ;$$

its reciprocal is a sharp amplitude because the Inversion Criterion (Section 5) has the immediate corollary that existence of this reciprocal is necessary and sufficient for global invertibility of the apparent solution. The notion of "small" amplitude is not here germane. Observe that the Riemann representation (Appendix A) offers the means of writing down formulae representing the functional in terms of the incidence data. They are too complicated, however, to be of much practical use. (The situation is similar for other conservation laws of the class typified by (1), (2): A "sharp amplitude" can be defined constructively, but to derive realistic bounds, is another matter.)

More light can be shed on the issue by applications of the theory [16]. For incident waves which are simple-harmonic in time, far from shore, the amplitude bound is there shown to be unity. For waves of much more general time-dependence, it is shown to amount essentially to a restriction on the high-frequency part of the amplitude 'spectrum'. The Invertibility Corollary (Section 5) shows, incidentally, that the strong solution of the conservation laws can be extended beyond $\sigma = 1$, once the Inversion Criterion is satisfied, and this opens the way for a consideration of the asymptotics of the solution for $\sigma >> 1$. It shows [16] that such solutions describe properties of water waves out to quite unexpectedly large distances from shore. What has to be small turns out [16] to be the amplitude measure appropriate to strong solutions far from shore. The asymptotic considerations [16] also indicate clearly how the solution structure far from the singular line remains controlled by the solution properties in the critical region near the singular line, which has been the domain of the present study.

APPENDIX A. PROOF OF EXISTENCE

It is marginally simpler to focus attention first on $Y(\sigma,\lambda) = y(\alpha,\beta)$ and to extend the incidence data 0-compatibly to $[-\varepsilon_1, \bar{\alpha} + \varepsilon_2]$ with arbitrarily small $\varepsilon_1, \varepsilon_2 > 0$ and $\hat{a}(\alpha) \equiv 0$ for $\alpha < 0$ in order to avoid superfluous limits at corners of the apparent domain (19). Then y is defined on the incidence boundary I (Fig. 6), on which $\beta = 1$, as

$$y(\alpha,1) = \hat{a}(\alpha) + \frac{3}{2}\int_0^{\alpha} \frac{\hat{a}(\alpha')}{\alpha' + 1}\, d\alpha' = \hat{y}(\alpha) \tag{A1}$$

which is also in $S_L[0,\bar{\alpha}']$ with $\hat{y}(0) = 0$ and $\bar{\alpha}' = \bar{\alpha} + \varepsilon_2$. Since $\varepsilon_1 > 0$, (13) is a regular differential equation on the subdomain $0 < \alpha < \bar{\alpha}'$, $2\varepsilon_1 < \beta < 1$ (Fig. 6) and the initial data (17) and incidence data (A1) pose for (13) a classical problem on this subdomain, which is known [22] to be well-posed and to preserve the regularity of the data. In particular, the "output function" $y(\bar{\alpha}',\beta)$, which describes the reflected wave issuing from the subdomain, is in $S_L[2\varepsilon_1, 1]$.

To solve the apparent problem in the rest of the domain, where $\beta < 0$ (Fig. 6) requires appeal to the shore condition (15), and the device for simplifying the proof is to postpone that. Instead, start by extending the output function $y(\bar{\alpha}',\beta)$ just mentioned to any function

$$\bar{y}(\beta) \in S_L[\varepsilon_3 - \bar{\alpha}', 1]$$

with arbitrary $\varepsilon_3 > 0$ and $\bar{y}(\beta) \equiv y(\bar{\alpha}',\beta)$ for $\beta > 0$. As long as $\alpha + \beta = \sigma > \varepsilon_3$, (13) remains regular and the data $\hat{y}(\alpha)$, $\bar{y}(\beta)$ set a classical, characteristic boundary value problem, which is well-posed [22] and of which the solution $y(\xi,\eta)$ has the Riemann representation [22]

$$y(\xi,\eta) = \hat{y}(\xi) + \bar{y}(\eta) - \hat{y}(\bar{\alpha}')R(\zeta_E) + \int_{\xi}^{\bar{\alpha}'} \hat{y}(\alpha)\frac{\partial R_I}{\partial \alpha}\, d\alpha + \int_{\eta}^{1} \bar{y}(\beta)\frac{\partial R_N}{\partial \beta}\, d\beta \tag{A2}$$

for $\xi + \eta = s > \varepsilon_3$, where

$$R(\zeta) = F\left(-\frac{1}{2}, \frac{3}{2}; 1; -\zeta\right), \qquad \zeta = \frac{(\alpha - \xi)(\beta - \eta)}{(\alpha + \beta)s} \tag{A3}$$

is the Riemann Function [22] of (13) and

$$\zeta_E = \frac{(\bar{\alpha}' - \xi)(1 - \eta)}{(1 + \bar{\alpha}')s} \quad \text{(at } E', \text{ Fig. 6)},$$

$$R_I = R(\zeta_I), \qquad \zeta_I = \frac{(\alpha - \xi)(1 - \eta)}{(1 + \alpha)s} \quad \text{(on } I\text{)},$$

$$R_N = R(\zeta_N), \qquad \zeta_N = \frac{(\bar{\alpha}' - \xi)(\beta - \eta)}{(\bar{\alpha}' + \beta)s} \quad \text{(on } N\text{)},$$

These values of ζ are > 0 and also bounded, as long as $s > \varepsilon_3 > 0$, and $R(\zeta)$ is then analytic, so that (A2) shows explicitly how $y(\xi,\eta) \in S_L(K)$ on any subdomain K where $s > \varepsilon_3$. And of course, (A2) agrees with the solution mentioned earlier on the subdomain $\beta > 0$.

If the restriction $s > \varepsilon_3 > 0$ is abandoned, however, then $\zeta \to \infty$ as $s = \xi + \eta \to 0$ (except on BJ and BH', Fig. 6) and

$$\frac{\partial\zeta}{\partial\alpha} = \frac{(\beta - \eta)(\beta + \xi)}{(\alpha + \beta)^2 s}, \qquad \frac{\partial\zeta}{\partial\beta} = \frac{(\alpha - \xi)(\alpha + \eta)}{(\alpha + \beta)^2 s} \tag{A4}$$

tend to ∞ similarly, and [26]

$$\zeta^{-1/2}R(\zeta) \to 4/\pi \qquad \zeta^{1/2}R'(\zeta) \to 2/\pi , \tag{A5}$$

and the representation (A2) of $y(\xi,\eta)$ fails. On the other hand,

$$\lim_{s\to 0} (s\zeta) = (\alpha + \eta)(\beta - \eta)/(\alpha + \beta) = \zeta_0 ,$$

$$\lim_{s\to 0} [s^{1/2}R(\zeta)] = R_0 = 4\zeta_0^{1/2}/\pi \qquad \lim_{s\to 0} (s^{-1/2}R'(\zeta)) = R_0' = 2\zeta_0^{-1/2}/\pi \tag{A6}$$

are defined and therefore, the 'quasi-Riemann' representation of the longshore mass-flow rate $\ell = (\alpha + \beta)^{1/2}y(\alpha,\beta)$, which (A2) for $s > \varepsilon_3$ shows to be

$$\ell(\xi,\eta) = s^{1/2}[\hat{y}(\xi) + \bar{y}(\eta) - \hat{y}(\bar{\alpha}')R(\zeta_E)]$$

$$+ \int_{\xi}^{\bar{\alpha}'} \hat{y}(\alpha)s^{1/2} \frac{\partial\zeta_I}{\partial\alpha} R'(\zeta_I)d\alpha + \int_{\eta}^{1} \bar{y}(\beta)s^{1/2} \frac{\partial\zeta_N}{\partial\beta} R'(\zeta_N)d\beta \;, \qquad \text{(A7)}$$

does have a continuous extension to the singular line $s = 0$ for $\eta > -\bar{\alpha}$.

Observe now that a function $y(\xi,\eta)$ has been displayed in (A2) which satisfies (13) for $\xi + \eta > 0$ and satisfies (17) and (18). By (A7), the shore condition (15) then amounts to the limit-integral equation

$$\lim_{s\to 0} \int_{\eta}^{1} \bar{y}(\beta)s^{1/2} \frac{\partial\zeta_N}{\partial\beta} R'(\zeta_N)d\beta$$

$$= \hat{y}(\bar{\alpha}') \lim_{s\to 0} [s^{1/2}R(\zeta_E)] - \lim_{s\to 0} \int_{\xi}^{\bar{\alpha}} \hat{y}(\alpha)s^{1/2} \frac{\partial\zeta_I}{\partial\alpha} R'(\zeta_I)d\alpha \qquad \text{(A8)}$$

for the output function $\bar{y}(\beta)$ on $[-\bar{\alpha},0]$.

Now, if (A8) really had a solution $\bar{y}(\beta)$ in S_L, then not only would existence be established, but a slightly better description [26] of $R(\zeta)$ than (A5) would also show readily that $\lim \int = \int \lim$ in (A8), which would then be an Abel equation,

$$(\bar{\alpha}'+\eta)^{3/2} \int_{\eta}^{1} (\beta-\eta)^{-1/2}(\bar{\alpha}'+\beta)^{-3/2}\, \bar{y}(\beta)d\beta$$

$$= 2\left[\frac{(\bar{\alpha}'+\eta)(1-\eta)}{(\bar{\alpha}'+1)}\right]^{1/2} \hat{y}(\bar{\alpha}') - (1-\eta)^{3/2} \int_{-\eta}^{\bar{\alpha}'} (\alpha+\eta)^{-1/2}(1+\alpha)^{-3/2}\, \hat{y}(\alpha)d\alpha$$

$$= (\bar{\alpha}'+\eta)^{3/2}\, g(\eta) \;, \qquad \text{(A9)}$$

say, with $g(\eta)$ known in terms of the incidence data (A1).

That the distributional solution of (A9) is

$$\bar{y}(\beta) = \frac{2}{\pi} (\bar{\alpha}'+\beta)^{3/2} \frac{d^2}{d\beta^2} \int_{\beta}^{1} (\beta-\mu)^{1/2}\, g(\mu)d\mu \qquad \text{(A10)}$$

is seen readily [27]. That it is also a classical solution, is best shown by a somewhat laborious, direct calculation of the righthand side, using the S_L-property of $\hat{y}(\alpha)$ to justify interchanges of operations by standard arguments in order to check that this expression is defined and integrable and satisfies (A9). Uniqueness then follows from the Fredholm Alternative. By carrying this calculation considerably further along the same line, it is also found [19] that the righthand side of (A10) is, in fact, in $S_L[-\bar{\alpha}, 1]$, whence it satisfies also (A8) and is therefore the function that should have been chosen for $\bar{y}(\beta)$, in the first place, to obtain the representation (A2).

Since it has already been noted that $y(\xi,\eta)$ has then the regularity claimed in the theorem and since its uniqueness and continuous dependence on the data for $\xi + \eta > \epsilon_3$ is classical [22] $\forall \epsilon_3 > 0$, and since $\ell(\xi,\eta)$ cannot have two extensions by continuity to $\xi + \eta = 0$, the proof of the theorem is complete for Y.

That for Z is closely analogous; the difference is that the Riemann function of (14) is more singular [26] as $\zeta \to \infty$, so that the parallel argument yields an extension to $\xi + \eta = 0$ only for the shoreward mass-flow rate $\sigma^{3/2}Z$.

The relation of the apparent problem to the Abel equation (A9) shows, incidentally, why the function class S_L (Section 3) is the one yielding symmetry in the regularity of input $\hat{y}$ and output $\bar{y}$. (This also carries over clearly to other conservation laws for which the singularity in the canonical equations is a pole with half-integer residue.)

To obtain the Incidence Corollary (Section 3), it now suffices to observe that $\partial^n Y/\partial\lambda^n$ and $\partial^n Z/\partial\lambda^n$ also satisfy (13) and (14), respectively, because the coefficient functions of those equations are independent of λ, and n-compatible incidence data pose for those derivatives exactly the problem which the theorem treats for Y and Z. The other corollary then follows by recursion from inspection of the integral for $\partial^{n-1}Y/\partial\lambda^{n-1}$ in terms of $\partial^n Y/\partial\lambda^n \in S_L(K)$, etc.

Proof of Shen's Lemma [20]. If

$$\frac{\partial b}{\partial\alpha} + \frac{3/2}{\alpha+\beta}\, a = f \quad \text{and} \quad \frac{\partial a}{\partial\beta} + \frac{3/2}{\alpha+\beta}\, b = -g \ ,$$

say, then the apparent equations (13), (14) for Y and Z read

$$\partial f/\partial\beta = -\frac{3}{2}(\alpha+\beta)^{-1} g, \qquad \partial g/\partial\alpha = -\frac{3}{2}(\alpha+\beta)^{-1} f \ .$$

Comparison with (7)-(10) shows $E(\sigma,\lambda) = f - g$ and $H(\sigma,\lambda) = f + g$ to satisfy (13) and (14), respectively, with zero initial and incidence data, by the definition of 1-compatibility. By (15),

$$\sigma^{3/2}\partial Y/\partial\sigma = \sigma\partial(\sigma^{1/2}Y)/\partial\sigma - \sigma^{1/2}Y/2 \to 0$$

as $\sigma \to 0$, and by the Incidence Corollary, also $\sigma^{3/2}\partial Z/\partial\lambda \to 0$, whence

$$\sigma^{3/2}H = \frac{3}{2}\sigma^{1/2}Y + 2\sigma^{3/2}(\partial Z/\partial\lambda - \partial Y/\partial\sigma) \to 0 ,$$

as well. That establishes for H all the assumptions of the theorem for Z, and by the uniqueness, $H \equiv 0$ on the apparent domain. But then, $E = 2f = 2g$ and $\partial E/\partial\lambda = 3\sigma^{-1}H/2 \equiv 0$, whence $E \equiv 0$ as well, because it vanishes initially and on the incidence line I (Fig. 3).

General Initial Conditions. For the wave equations (13) and (14), there is no significant loss of generality, and a considerable gain in lucidity of notation, in envisaging initial data of Cauchy type on the line $\lambda = -1$ (Fig. 2). The issue to be addressed now concerns the restrictions on general Cauchy data there arising from the singular nature of the conservation laws (1), (2) for solutions of the type obtained in the Incidence Theorem.

It will have been observed that the formulation of Section 2 sets, not independent problems for (13) and for (14), but a related problem pair. This is achieved there, somewhat inconspicuously, by specifying in (18) just one function, $a(\alpha,1)$, from which more direct input data for Y or Z are to be generated by the canonical equations, according to the compatibility definition (Section 3). In the same way, Cauchy data for (13) and for (14) cannot be independent, if the apparent problem is to concern the conservation laws (1), (2). It is possible to prescribe Y and $\partial Y/\partial\lambda$ independently on the initial line $\lambda = -1$, but then no choice is left in the prescription of Z and $\partial Z/\partial$, and vice-versa. For nonlinear wave equations, such as (1), (2), it is always more natural, and promotes lucidity, to specify canonical variables and hence, it will be envisaged now that the acceleration measures (5), (6) are to be specified for $\lambda = -1$ as functions

$$a = A_0(\sigma), \qquad b = B_0(\sigma) ,$$

for $0 < \sigma < 1$, and that Y, $\partial Y/\partial\lambda$, Z and $\partial Z/\partial\lambda$ are there to be computed from A_0 and B_0 by (10) and by the canonical equations (11), (12).

To determine the type of functions that can be specified, it helps to direct attention first to the case of 0-compatibility and secondly, to begin by specifying $A_0(\sigma)$ and $B_0(\sigma)$ only for $0 < \varepsilon_3 < \sigma < 1$, i.e., on the line segment A'C (Fig. 7). That sets a classical problem [22] for the square domain A'M'CG' (Fig. 7), because (13) and (14) are there regular wave equations, and hence, this specification is equivalent [22] to that of a along M'C' and of b, along G'C or along A'M'. To obtain solutions in the class $S_L(K)$ of the Incidence Theorem (Section 3), it is clearly necessary that

$$A_0(\sigma), B_0(\sigma) \in S_L[\varepsilon_3, 1] \; \forall \; \varepsilon_3 > 0 \; . \tag{A11}$$

This is also sufficient to generate a unique solution [22] in the class S_L on the square A'M'CS', because the classical problem preserves the regularity of the data [22].

The apparent domain, however, has now been extended to $-(1 - \varepsilon_3)/2 < \alpha < \bar{\alpha}$, $\varepsilon_3 < \sigma < 1 + \bar{\alpha}$, and to obtain a solution in S_L on this domain, the condition

$$A_0(1) = \hat{a}(0) \tag{A12}$$

must also be imposed. It is necessary because a discontinuity of $a(\alpha,\beta)$ at C (Fig. 7) would "propagate" along $\alpha = 0$ and be reflected [1] from the singular line as a logarithmic singularity of $b(\alpha,\beta)$. It is also sufficient, because it assures that $a(\alpha,1) \in S_L[(\varepsilon_3 - 1)/2, \bar{\alpha}]$ all along the incidence characteristic $\beta = 1$.

The interesting part of the question, of course, is what conditions on $A_0(\sigma)$ and $B_0(\sigma)$ arise near $\sigma = 0$ when we now let $\varepsilon_3 \to 0$. From (10), (15) and (16), it is clear that the mass-flow rates

$$\sigma^{1/2}(A_0 - B_0) = L(\sigma) \quad \text{and} \quad \sigma^{3/2}(A_0 + B_0) = M(\sigma)$$

must tend to zero as $\sigma \to 0$. The proof of the Incidence Theorem, however, depends on interchanges of operations for which it is necessary and sufficient to appeal to (15) and (16) also in the sense that $\partial(\sigma^{1/2}Y)/\partial\lambda \to 0$ and $\partial(\sigma^{3/2}Z)/\partial\lambda \to 0$ as $\sigma \to 0$. To

$$L(\sigma) \to 0 \quad \text{and} \quad M(\sigma) \to 0 \quad \text{as} \quad \sigma \to 0$$

must therefore, by (11), (12), be added that

$$\sigma^{-1}M'(\sigma) \to 0 \quad \text{and} \quad \sigma L'(\sigma) \to 0 \quad \text{as} \quad \sigma \to 0 \; . \tag{A13}$$

The Incidence Theorem (Section 3) might suggest that the conditions so far listed be sufficient, but the Regularity Theorem (Section 4) shows that apparent solutions for 0-compatible data possess additional regularity near the singular line. For consistency with this, $L \to 0$, $M \to 0$ is not enough, but

$$\sigma^{\varepsilon-1/2}L(\sigma) \to 0 \quad \text{and} \quad \sigma^{\varepsilon-1/2}M(\sigma) \to 0 \quad \text{as} \quad \sigma \to 0, \quad \forall\ \varepsilon > 0 \tag{A14}$$

is sufficient.

For apparent solutions corresponding to higher degrees of compatibility, the conditions (A11)-(A14) must, of course, be extended to further derivatives in the way clearly indicated by the corollaries of Sections 3, 4 and the arguments just listed.

It should also be remarked that, to preserve the theorems of Sections 3-6 in the case of general initial data, (A11)-(A14) and their extensions to further derivatives must be added to the definition of compatibility (Section 3), lengthening it to a half-page of print... The consequent loss of lucidity in presentation and proofs is one reason for stripping the canonical example of conservation laws down to (17). It is reinforced by the observation that, while initial data play an important textbook role, the mathematically natural formulation for hyperbolic systems generates solutions by incident waves. From the point of view of the mathematical structure of such laws, initial data only inject a remnant of a wave process that should really have been described by the solution.

APPENDIX B. PROOF OF REGULARITY.

The structure of the proof will be clarified by focusing attention first on $Y(\sigma,\lambda) = y(\alpha,\beta)$ in the case of 0-compatible data on $[0,\bar{\alpha}]$. For simplicity of notation also, (A2) may be used with $\varepsilon_2 = 0$, $\bar{\alpha}' = \bar{\alpha}$, so that it reads

$$y(\xi,\eta) = \hat{y}(\xi) + \bar{y}(\eta) - \hat{y}(\bar{\alpha})R(\zeta_E) + \int_\xi^{\bar{\alpha}} \hat{y}(\alpha) \frac{\partial}{\partial\alpha} R(\zeta_I)d\alpha + \int_\eta^1 \bar{y}(\beta) \frac{\partial}{\partial\beta} R(\zeta_N)d\beta \tag{B1}$$

with the Riemann function (A3) and

$$\zeta_E = \frac{(\bar{\alpha} - \xi)(1 - \eta)}{(\alpha + 1)s} \quad \text{at} \quad E \quad \text{(Fig. 8)} ,$$

$$\zeta_I = \frac{(\alpha - \xi)(1 - \eta)}{(\alpha + 1)s} \quad \text{on} \quad I ,$$

$$\zeta_N = \frac{(\bar{\alpha} - \xi)(\beta - \eta)}{(\bar{\alpha} + \beta)s} \quad \text{on} \quad N ,$$

and again

$$s = \xi + \eta .$$

It is helpful to consider $s^{1/2}R(\zeta) = Q(s\zeta; s)$ as a function of $s\zeta$, and of s as parameter, and to relieve the notation of the reference to the residual dependence on s by writing

$$s^{1/2}R(\zeta) = Q(s\zeta) . \tag{B2}$$

Then by (A5),

$$\lim_{s\to 0} [s^{1/2}R(\zeta)] = Q(\lim s\zeta) , \tag{B3}$$

and in the first integral of (B1), where $\beta = 1$,

$$s\zeta_I = (\alpha-\xi)(\bar{\sigma}-\eta)/(\alpha+\bar{\sigma}) = \tau(\alpha), \quad \lim_{\xi\to-\eta} (s\zeta_I) = \tau_0(\alpha) , \tag{B4}$$

while in the second, where $\alpha = \bar{\alpha}$,

$$s\zeta_N = (\bar{\alpha}-\xi)(\beta-\eta)/(\bar{\alpha}+\beta) = \theta(\beta), \quad \lim_{\xi\to-\eta} (s\zeta_N) = \theta_0(\beta) . \tag{B5}$$

As shown in Appendix A, the output function

$$\bar{y}(\beta) = y(\bar{\alpha},\beta) \in S_L[-\bar{\alpha},1] \,, \tag{B6}$$

and $s^{1/2}y(\xi,\eta) \to 0$ as $s \to 0$ in such a way that the limit commutes with the integrals to give (A8) in the form

$$0 = -\hat{y}(\bar{\alpha})Q(\tau_0(\bar{\alpha})) + \int_{-\eta}^{\bar{\alpha}} \hat{y}(\alpha) \frac{\partial}{\partial\alpha} Q(\tau_0)d\alpha + \int_{\eta}^{1} \bar{y}(\beta) \frac{\partial}{\partial\beta} Q(\theta_0)d\beta \,. \tag{B7}$$

To explore now just how $s^{1/2}y(\xi,\eta) \to 0$, multiply both sides of (B1) by $s^{1/2}$ and then subtract (B7):

$$\begin{aligned} s^{1/2}y(\xi,\eta) = {} & \int_{\eta}^{1} \bar{y}(\beta) \frac{\partial}{\partial\beta} [Q(\theta)-Q(\theta_0)]d\beta + \int_{\xi}^{\bar{\alpha}} [\hat{y}(\alpha)-\hat{y}(\bar{\alpha})] \frac{\partial}{\partial\alpha} Q(\tau)d\alpha \\ & + s^{1/2}[\hat{y}(\xi) + \bar{y}(\eta) - \hat{y}(\bar{\alpha})] - \int_{-\eta}^{\bar{\alpha}} [\hat{y}(\alpha)-\hat{y}(\bar{\alpha})] \frac{\partial}{\partial\alpha} Q(\tau_0)d\alpha \end{aligned} \tag{B8}$$

because $R \equiv 1$ for $\alpha = \xi$ and for $\beta = \eta$, regardless of the value of s. This is a second quasi-Riemann representation of the longshore mass-flow rate $\sigma^{1/2}Y$, which is more delicate than (A7) and can be used to advantage, once the Incidence Theorem has established (B6). It may be noticed that (B8) compares the values of $\sigma^{1/2}Y$ at P and P_0 (Fig. 8), so that it contains information only on the manner in which $s^{1/2}y(\xi,y) \to 0$ as $s = \xi + \eta \to 0$ at fixed η. However, if the representation were to be arranged so as to give information as $s = \xi + \eta \to 0$ at fixed ξ (i.e., vertically, in the Figure, instead of horizontally), then $\hat{y}$ and $\bar{y}$ would simply switch their roles; and if other approaches to the singular line were to be studied, then $\hat{y}$ and $\bar{y}$ would appear in quite similar ways in the representation, at a considerable expense in additional notation. What this remark serves, is to demonstrate that the information from (B8) is the general one, because $\bar{y}$ and $\hat{y}$ are in the same function class S_L and no more has been specified for either. If they were not both in S_L, on the other hand -- and there is some latitude in the choice of incidence theorems, if such asymmetry of input and output be accepted -- then the manner of approach to the singular line would make a difference to the result and the regularity theorem would be more complicated.

To deduce the present Regularity Theorem from (B8), only the integrals there need scrutiny, because $\hat{y}$ and $\bar{y}$ are bounded, by (A1) and (B6). The first integral in (B8) is

$$I_1 = \int_0^{\theta(1)} \bar{y}(\beta)[Q'(\theta) - Q'(\theta_0)d\theta_0/d\theta]d\theta$$

and from (B5),

$$\theta_0(\beta) = \theta(\beta)\left(1 + \frac{s}{\bar{\alpha}+\eta-s}\right) . \tag{B9}$$

The rapid variation of the Riemann function near $\beta = \eta$ (Fig. 8) prompts a split:

$$I_{11} = \int_0^{s^{1/2}} \bar{y}(\beta)\left[\{Q'(\theta) - Q'(\theta_0)\}\frac{d\theta_0}{d\theta} + \left(1 - \frac{d\theta_0}{d\theta}\right)Q'(\theta)\right]d\theta$$

$$= -\int_0^{s^{1/2}} \frac{\bar{y}(\beta)}{\bar{\alpha}+\eta-s}\left[sQ'(\theta) + (\bar{\alpha}+\eta)\int_\theta^{\theta_0} Q''(\tau)d\tau\right]d\theta$$

and since $Q'(\theta) = s^{-1/2}R'(\zeta)$ and $Q''(\theta) = s^{-3/2}R''(\zeta)$ and $\theta_0-\theta = s\theta/(\bar{\alpha}+\eta-s)$ and furthermore, $R'(\zeta)$ and $R''(\zeta)$ are analytic on $[0,\infty)$ and tend to zero as $\zeta \to \infty$,

$$|I_{11}| < s \operatorname{lub}\left|\frac{\bar{y}}{\bar{\alpha}+\eta-s}\right|\left\{\max|R'| + \operatorname{lub}\left|\frac{\bar{\alpha}+\eta}{\bar{\alpha}+\eta-s}\right|\max|R''|\right\}$$

where the lub's are over the segment of characteristic $\alpha = \bar{\alpha}$ on which $0 < \theta < s^{1/2}$. But there, only $\bar{y}$ varies with $\theta(\beta)$ and it is bounded, by (B7). For any given η, moreover, a positive lower bound on $\bar{\alpha} + \eta - s$ is guaranteed by an appropriate choice of $\bar{\alpha}$, which is quite unrestricted. Hence, $s^{-1}I_{11}$ is bounded as $s \to 0$.

The rest of I_1 is

$$I_{12} = \int_{s^{1/2}}^{\theta(1)} \bar{y}(\beta)\left[Q'(\theta) - Q'(\theta_0)\frac{d\theta_0}{d\theta}\right]d\theta ,$$

and for small s and $\theta > s^{1/2}$, $\zeta = \theta/s >> 1$ and by (A6),

$$Q'(\theta) = s^{-1/2}R'(\zeta) \sim \frac{2}{\pi}\,\theta^{-1/2}[1 + 0(\frac{s}{\theta})] ,$$

so that

$$\frac{Q'(\theta_0)}{Q'(\theta)}\,\frac{d\theta_0}{d\theta} - 1 = (\frac{\theta_0}{\theta})^{1/2}\,[1 + 0(\frac{s}{\theta})] - 1 = \frac{s}{2\bar{\alpha} + 2\eta}\,(1 + 0(s^{1/2})) ,$$

by (B9), and

$$I_{12} = \frac{-s}{2\bar{\alpha} + 2\eta}\int_{Q(s^{1/2})}^{Q(\theta(1))} \bar{y}(\beta)[1 + 0(s^{1/2})]dQ .$$

Since $\pi Q(\theta(1)) \sim 4[\theta_0(1)]^{1/2}$ and $\pi Q(s^{1/2}) \sim 4s^{1/4}$, by (A6), also $s^{-1}I_{12}$ is bounded as $s \to 0$ when $\bar{\alpha}$ is chosen appropriately.

To estimate the other two integrals in (B6), note that on I, between J_0 and E (Fig. 8), where $\beta = 1$,

$$\tau(\alpha) = > 0, \qquad \tau_0(\alpha) = \tau(\alpha) + s(1-\eta)/(\alpha+1) > \tau(\alpha) ,$$

by (B4), and inversely,

$$\alpha = \frac{(\tau + \xi) - \xi\eta}{1 - \eta - \tau} = \alpha(\tau) ,$$

$$= \frac{(\tau_0 - \eta) + \eta^2}{1 - \eta - \tau_0} = \tilde{\alpha}(\tau_0) = \alpha(\tau_0) - s\,\frac{1 - \eta}{1 - \eta - \tau_0} . \qquad \text{(B10)}$$

The other two integrals in (B8) together are therefore

$$I_2 = \int_0^{\tau(\bar{\alpha})} [\hat{y}(\alpha(v)) - \hat{y}(\tilde{\alpha}(v))]Q'(v)dv - \int_{\tau(\bar{\alpha})}^{\tau_0(\bar{\alpha})} [\hat{y}(\tilde{\alpha}(\tau_0)) - \hat{y}(\bar{\alpha})]Q'(\tau_0)d\tau_0 .$$

For the first of these new integrals, $0 < v < \tau(\bar{\alpha})$ makes, by (A1),

$$|\hat{y}(\alpha) - \hat{y}(\tilde{\alpha})| < m_1|\alpha(v) - \tilde{\alpha}(v)|^{\gamma} = m_1 s^{\gamma}|\frac{1-\eta}{1-\eta-v}|^{\gamma} = m_1 s^{\gamma}|1 - \frac{v}{1-\eta}|^{-\gamma}$$

$$< m_1 s^{\gamma}|1 - \frac{\tau(\bar{\alpha})}{1-\eta}|^{-\gamma} = m_1 s^{\gamma}|\frac{\bar{\alpha}+1}{1+\xi}|^{\gamma} < m_2 s^{\gamma}$$

for every $\gamma < 1/2$ and numbers m_1, m_2 independent of α and s. For the other integral, since $d\tau_0/d\alpha > 0$ and $\alpha(\tau(\bar{\alpha})) = \tilde{\alpha}(\tau_0(\bar{\alpha})) = \bar{\alpha}$, it is seen that $\tau(\bar{\alpha}) < \tau_0 < \tau_0(\bar{\alpha})$ implies

$$\bar{\alpha} > \tilde{\alpha}(\tau_0) > \tilde{\alpha}(\tau(\bar{\alpha})) = \bar{\alpha} - s(1-\eta)/(1-\eta-\tau(\bar{\alpha}))$$

and

$$|\hat{y}(\tilde{\alpha}(\tau_0)) - \hat{y}(\bar{\alpha})| < m_1|\bar{\alpha} - \tilde{\alpha}(\tau_0)|^{\gamma}$$

$$< m_1|\bar{\alpha} - \tilde{\alpha}(\tau(\bar{\alpha}))|^{\gamma} = m_1 s^{\gamma}|\frac{1-\eta}{1-\eta-\tau(\bar{\alpha})}| = m_1 s^{\gamma}|\frac{\bar{\alpha}+1}{1+\xi}|^{\gamma} < m_2 s^{\gamma} ,$$

by (B10) and (B4). Therefore

$$|I_2| < m_2 s^{\gamma}\{|s^{1/2}R(\frac{\tau(\bar{\alpha})}{s}) - s^{1/2}| + |s^{1/2}R(\frac{\tau_0(\bar{\alpha})}{s}) - s^{1/2}R(\frac{\tau(\bar{\alpha})}{s})|\}$$

and by (A5), $R(\tau(\bar{\alpha})/s) = O(s^{-1/2})$, but

$$R(\frac{\tau_0(\bar{\alpha})}{s}) - R(\frac{\tau(\bar{\alpha})}{s}) \sim (4/\pi)s^{-1/2}\{[\tau_0(\bar{\alpha})]^{1/2} - [\tau(\bar{\alpha})]^{1/2}\} = O(s^{1/2}) ,$$

so that $|I_2| < m_3 s^{\gamma}$. Hence, (B8) has been shown to imply that

$$s^{1/2-\gamma}y(\xi,\eta)$$

is bounded independently of ξ for all sufficiently small $s > 0$ and for all $\gamma < 1/2$.

If the incidence data are n-compatible, then the same proof applies to $\partial^n Y/\partial\lambda^n$, by the Incidence Corollary (Section 3). The proof for $\partial^n Z/\partial\lambda^n$ is analogous: the only difference is that the Riemann function of (14) has a different branch point at ∞.

APPENDIX C. PROOFS OF REGULARITY COROLLARIES

R1. By Shen's Lemma (Section 3), the canonical equation (11) assures

$$\frac{\partial}{\partial\sigma}\left(\sigma^{3/2}\frac{\partial^n Z}{\partial\lambda^n}\right) = \sigma^{3/2}\frac{\partial^{n+1}Y}{\partial\lambda^{n+1}} .$$

By the Incidence Corollary (Section 3), $\sigma^{3/2}\partial^n Z/\partial\lambda^n \to 0$ as $\sigma \to 0$, so that

$$\sigma^{\varepsilon-1}\frac{\partial^n Z(\sigma\lambda)}{\partial\lambda^n} = \int_0^1 v^{(3-2\varepsilon)/2}(v\sigma)^{\varepsilon}\frac{\partial^{n+1}}{\partial\lambda^{n+1}}Y(v\sigma,\lambda)\,dv ,$$

and by the Regularity Theorem $\sigma^{\varepsilon}\partial^{n+1}Y(\sigma,\lambda)/\partial\lambda^{n+1} \to 0$ as $\sigma \to 0$.

Admissibility Theorem. For any fixed $\sigma_0 \in (0,1]$, (10) and the Incidence Theorem assure existence of $T(\sigma_0,\lambda)$ for $\lambda \in [0,2\bar{\alpha} - \sigma_0]$. Again by (10), for $\sigma > 0$,

$$T(\sigma,\lambda) - T(\sigma_0,\lambda) = \int_{\sigma_0}^{\sigma} \tau^{\varepsilon-1}Z(\tau,\lambda)\tau^{-\varepsilon-1/2}d\tau ,$$

and by Corollary R1 with $0 < \varepsilon < \frac{1}{2}$, the integral tends to a limit as $\sigma \to 0$ for fixed λ.

R1a. It has been shown in [15, Section X] how admissibility of the apparent solution implies the assumptions of Shen's [17, 21] proof with the exception only of integrability of $\sigma^{-1/2}Y(\sigma,\lambda)$ with respect to σ up to $\sigma = 0$ for fixed λ. That integrability follows from the Regularity Theorem for $n = 0$ and, e.g., $\varepsilon = 1/4$.

R2. By Shen's Lemma (Section 3) and (12),

$$\sigma^{\varepsilon-1}\partial Z/\partial\lambda = \sigma^{\varepsilon+1/2}\partial(\sigma^{-3/2}Y)/\partial\sigma ,$$

and the lefthand side tends to zero with σ for $0 < \varepsilon < 5/2$, by Corollary R1. For $0 < \sigma_0 < 1$, therefore,

$$\sigma^{-3/2}Y(\sigma,\lambda) = \sigma_0^{-3/2}Y(\sigma_0,\lambda) + \int_{\sigma_0}^{\sigma}\tau^{\varepsilon-1}\frac{\partial}{\partial\lambda}Z(\tau,\lambda)\frac{d\tau}{\tau^{\varepsilon+1/2}}$$

and for $\varepsilon = 1/4$, e.g., this integral tends to a limit as $\sigma \to 0$.

Admissibility Corollary. The Incidence and Admissibility Theorems define a function $T(\sigma,\lambda)$ on $[0,1] \times [0,2\bar{\alpha}]$. If that function were not continuous at $\sigma = 0$, $\lambda = \lambda_1$ for some $\lambda_1 \in [0,2\bar{\alpha})$, then the limit of $\partial T/\partial\lambda$ could not there exist, contrary to Corollary R2.

R3. By Shen's Lemma (Section 3) and (11), (16),

$$\sigma^{3/2}Z(\sigma,\lambda) = \int_0^\sigma \tau^{3/2} \frac{\partial}{\partial\lambda} Y(\tau,\lambda)d\tau \ ,$$

for $0 < \sigma < 1$, so that

$$\sigma^{-5/2}Z(\sigma,\lambda) = \int_0^1 v^3(v\sigma)^{-3/2} \frac{\partial}{\partial\lambda} Y(v\sigma,\lambda)dv \ .$$

By Shen's Lemma and (12), in turn,

$$\sigma^{-3/2}\partial Y/\partial\lambda = \sigma_0^{-3/2}\partial Y/\partial\lambda\Big|_{\sigma_0,\lambda} + \int_{\sigma_0}^{\sigma} \tau^{\varepsilon-1} \frac{\partial^2}{\partial\lambda^2} Z(\tau,\lambda) \frac{d\tau}{\tau^{\varepsilon+1/2}}$$

for $0 < \sigma_0 < 1$, and this tends to a limit as $\sigma \to 0$ for fixed λ, by Corollary R1.

R3a. The Incidence Corollary (Section 3) and Corollary R2 define a function $\sigma^{-3/2}Y(\sigma,\lambda) = H(\sigma,\lambda)$ on $[0,1] \times [0,2\bar{\alpha} - \sigma]$, continuous for $\sigma > 0$. If $H(0,\lambda_1)$ were not continuous at some λ_1, then $\partial H/\partial\lambda$ could not both (a) be continuous for $\sigma > 0$ and (b) have a limit as $\sigma \to 0$ for $\lambda = \lambda_1$. The Incidence Corollary for $n > 1$ assures (a), however, and (b) follows from the analog of Corollary R3 concerning $\sigma^{-3/2}\partial Y/\partial\lambda$ available for 3-compatible data because $\partial Y/\partial\lambda$ satisfies the Incidence and Regularity Theorems with $n = 0$.

A parallel argument for Z and $\partial Z/\partial\lambda$ proves Corollary R4.

APPENDIX D. PROOF OF THE INVERTIBILITY LEMMA

Let D denote the apparent domain (19) and $\bar{D}$, its union with the singular line segment $\sigma = 0$, $0 \leq \lambda \leq 2\bar{\alpha}$.

Convexity Lemma. The assumptions of the Invertibility Lemma imply that any two distinct points I, II in $\bar{D}$ at which the clock-time T takes the same value, can be connected with $\bar{D}$ by a continuous curve $T(\sigma,\lambda) = \text{const.}$ on which σ varies strictly monotonely.

Given this lemma, if global invertibility of the characteristic transformation on $\bar{D}$ were false, i.e., if a pair of distinct points I, II in $\bar{D}$ could be found which map into the same (X,T), then a curve $T(\sigma,\lambda) = T_I = T_{II}$ would connect them and σ could serve as the parameter on it. On this curve, by (9),

$$d\lambda/d\sigma = (\partial t/\partial\alpha + \partial t/\partial\beta)/(\partial t/\partial\beta - \partial t/\partial\alpha) \qquad (D.1)$$

which is continuous in D, by (9), (10) and the Incidence Theorem and by the assumption (ii) of the Invertibility Lemma, and has the bound

$$|d\lambda/d\sigma| < 1 , \qquad (D.2)$$

by the assumption (i). By (4), (9) and (D.1), moreover

$$\frac{dX}{d\sigma} = \frac{\partial t}{\partial\alpha}\frac{\partial t}{\partial\beta} \bigg/ \left(\frac{\partial t}{\partial\beta} - \frac{\partial t}{\partial\alpha}\right) < 0$$

on this curve in D, by (i) and (ii). Hence, $X_{II} = X_I$ is incompatible with $T_{II} = T_I$ for distince points in $\bar{D}$.

To prove the Convexity Lemma, note that distinct points at which both $T_{II} = T_I$ and $\sigma_{II} = \sigma_I$ cannot occur in $\bar{D}$ because of the strict monotoneity of T in λ at fixed σ implied by (ii): the points may be labeled to that

$$0 \leq \sigma_I < \sigma_{II} .$$

Since (ii) shows the righthand side of (D.1) to be defined and continuous in D for 1-compatible data, a continuous curve $\Lambda : \lambda = \lambda(\sigma)$ with the local slope (D.1) may be traced from II in the sense of decreasing σ. On Λ, clock-time $T = \text{const} = T_{II}$, and the bound (D.2) shows that Λ must lead, if not (a) to a boundary point III of D at which $\sigma > 0$, then (b) to a point in $\bar{D}$ of the singular line $\sigma = 0$.

In case (b), Λ must intersect the line $\sigma = \text{const.} = \sigma_I > 0$ and that can be only at the point I, by (ii). The same conclusion applies in case (a), if $\sigma_{III} < \sigma_I$. The proof will therefore be complete, if (a) is shown incompatible with $\sigma_{III} > \sigma_I$: As Λ is traced in the sense of decreasing σ, (D.2) and (9) show α and β to decrease also, so that III must lie on the boundary segment CO of D (Fig. 3). Therefore, I can be reached from III by following first CO in the sense of decreasing σ until $\sigma = \sigma_I$ and then following the line $\sigma = \text{const.} = \sigma_I$ to I. On the way, T increases along CO, where $\alpha = 0$ and $u = 0$, by the initial condition (17), and so by (9), $\sigma + 2T = \text{const}$; and T increases also along the rest of the way, by (ii). Hence $T_I > T_{III} = T_{II}$, contrary to hypothesis.

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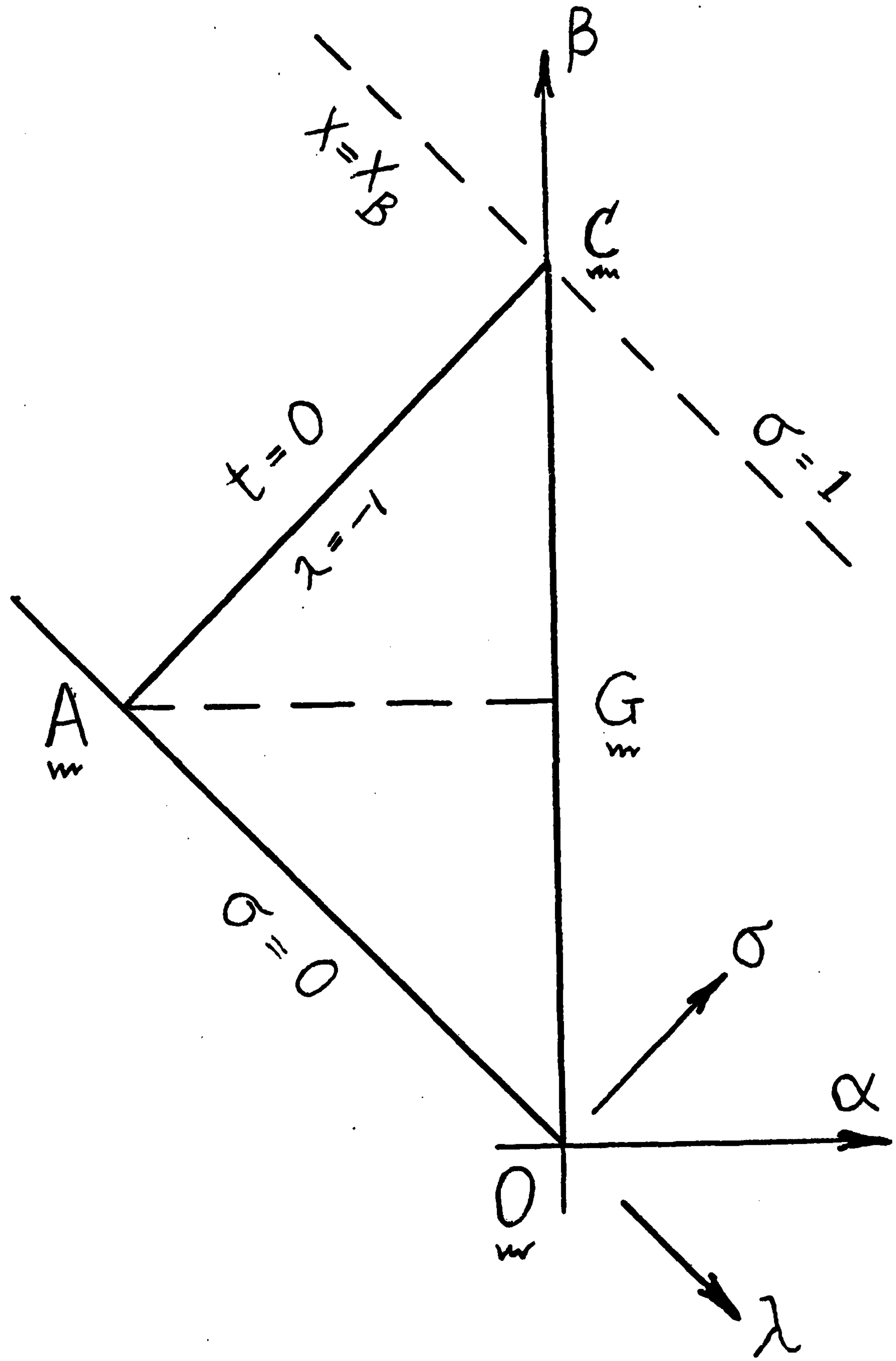
Figure 1



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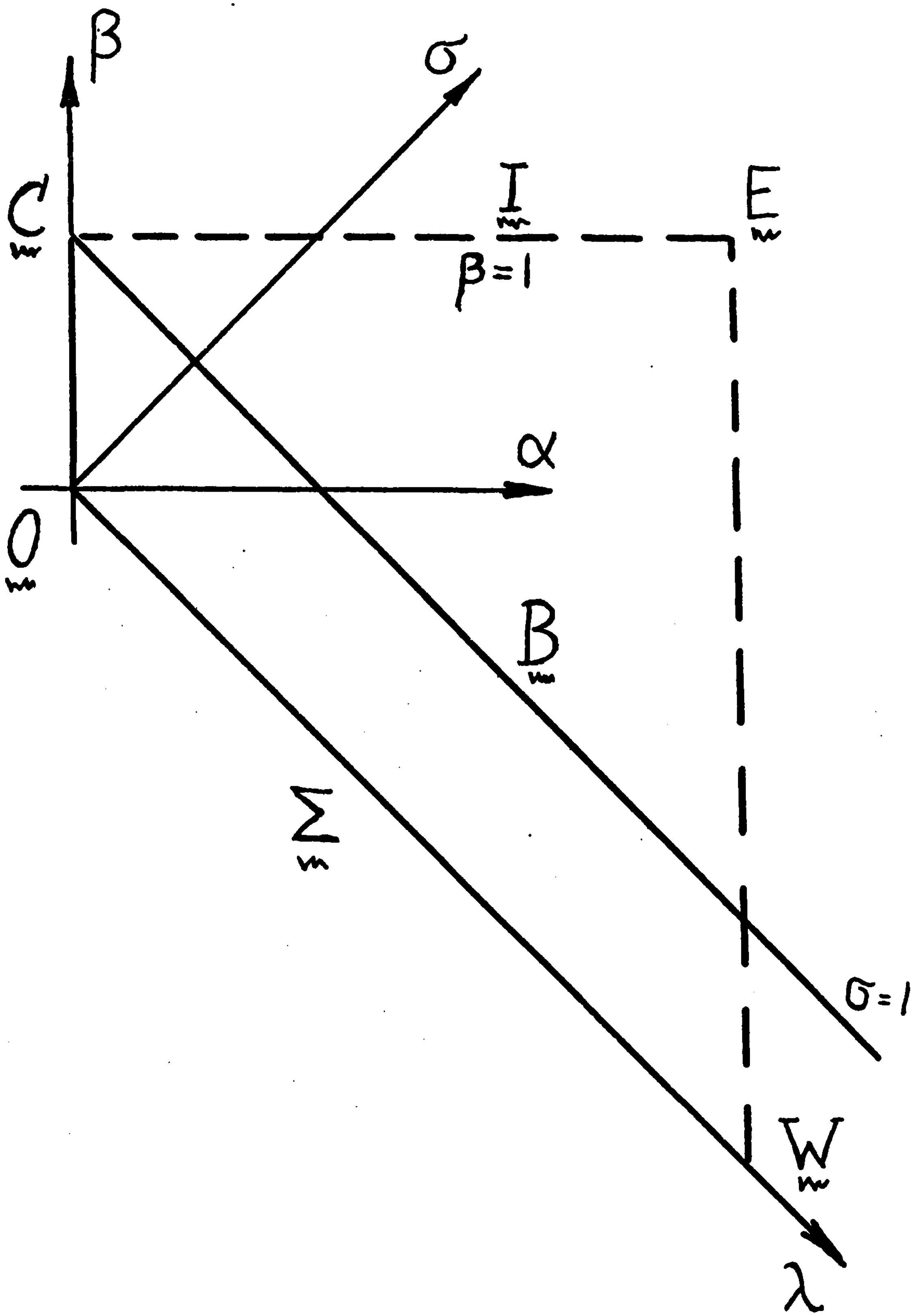

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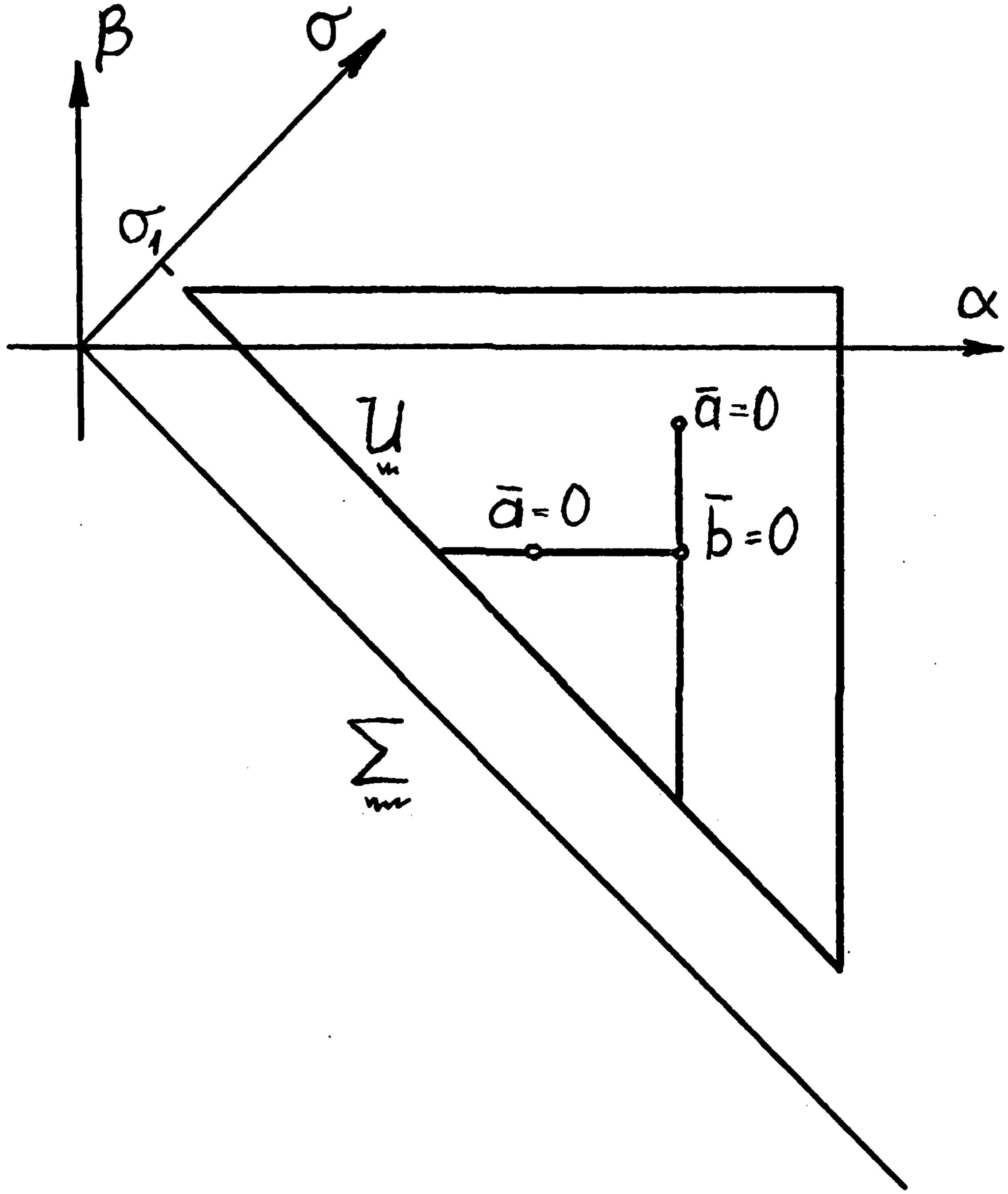

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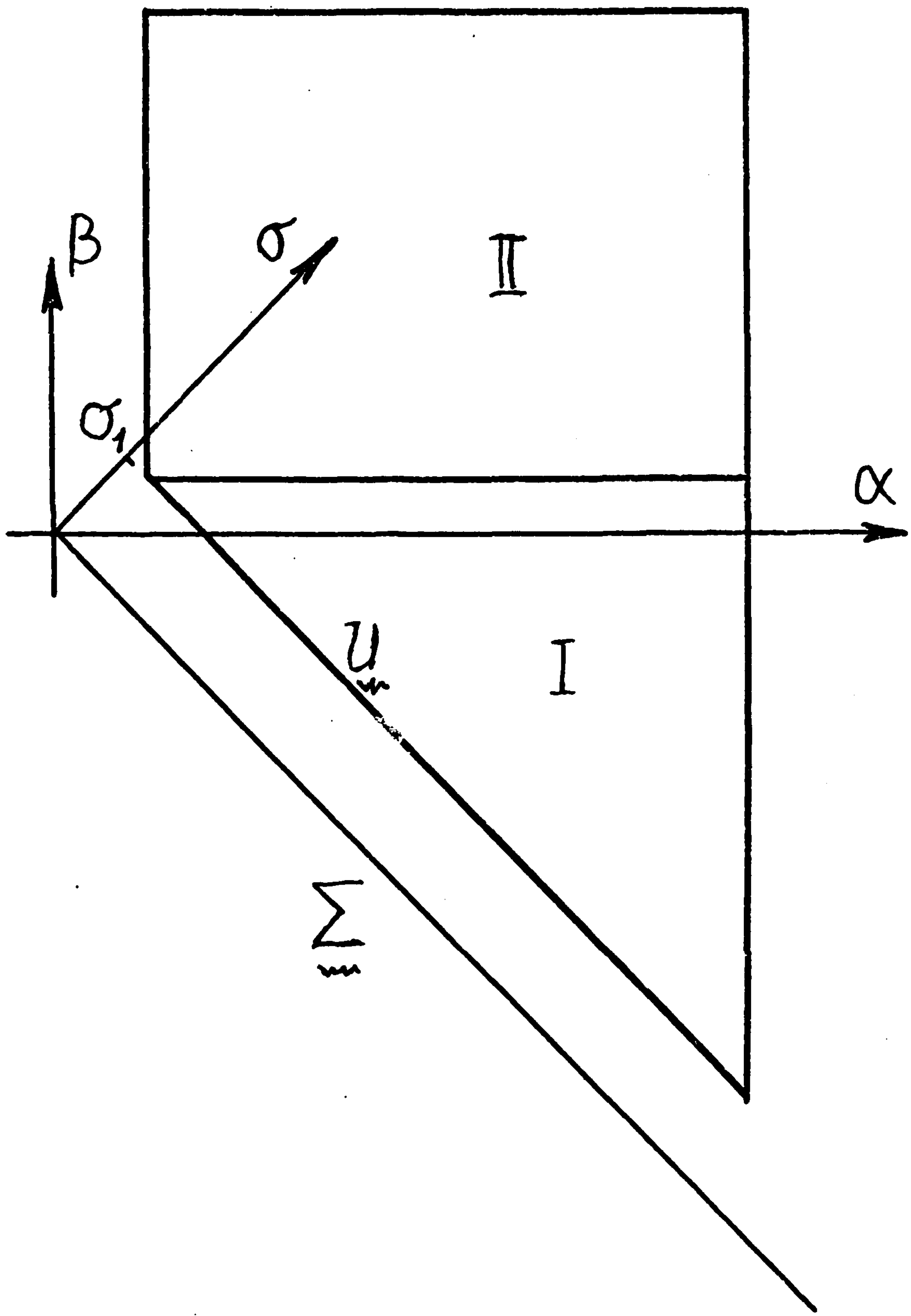

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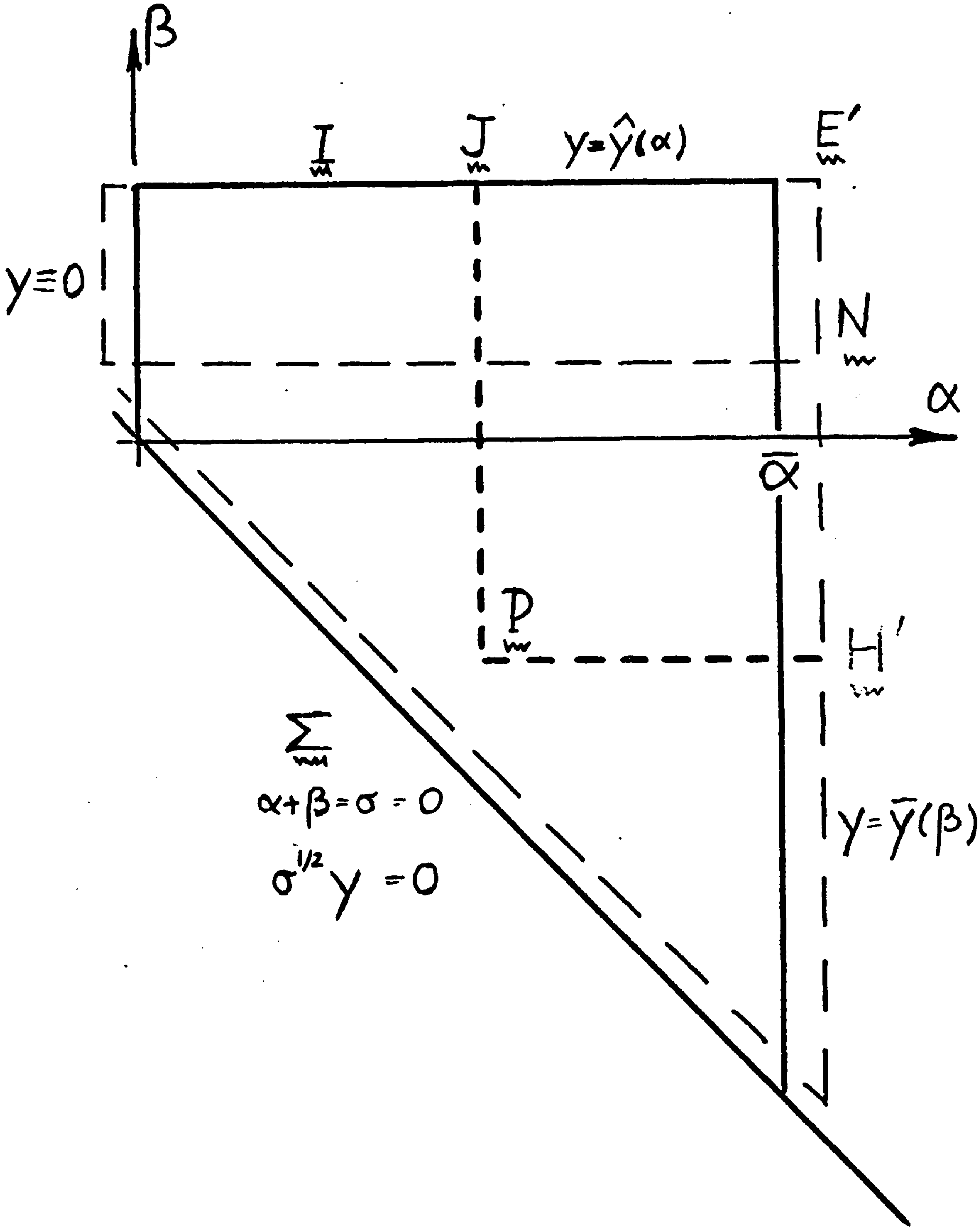

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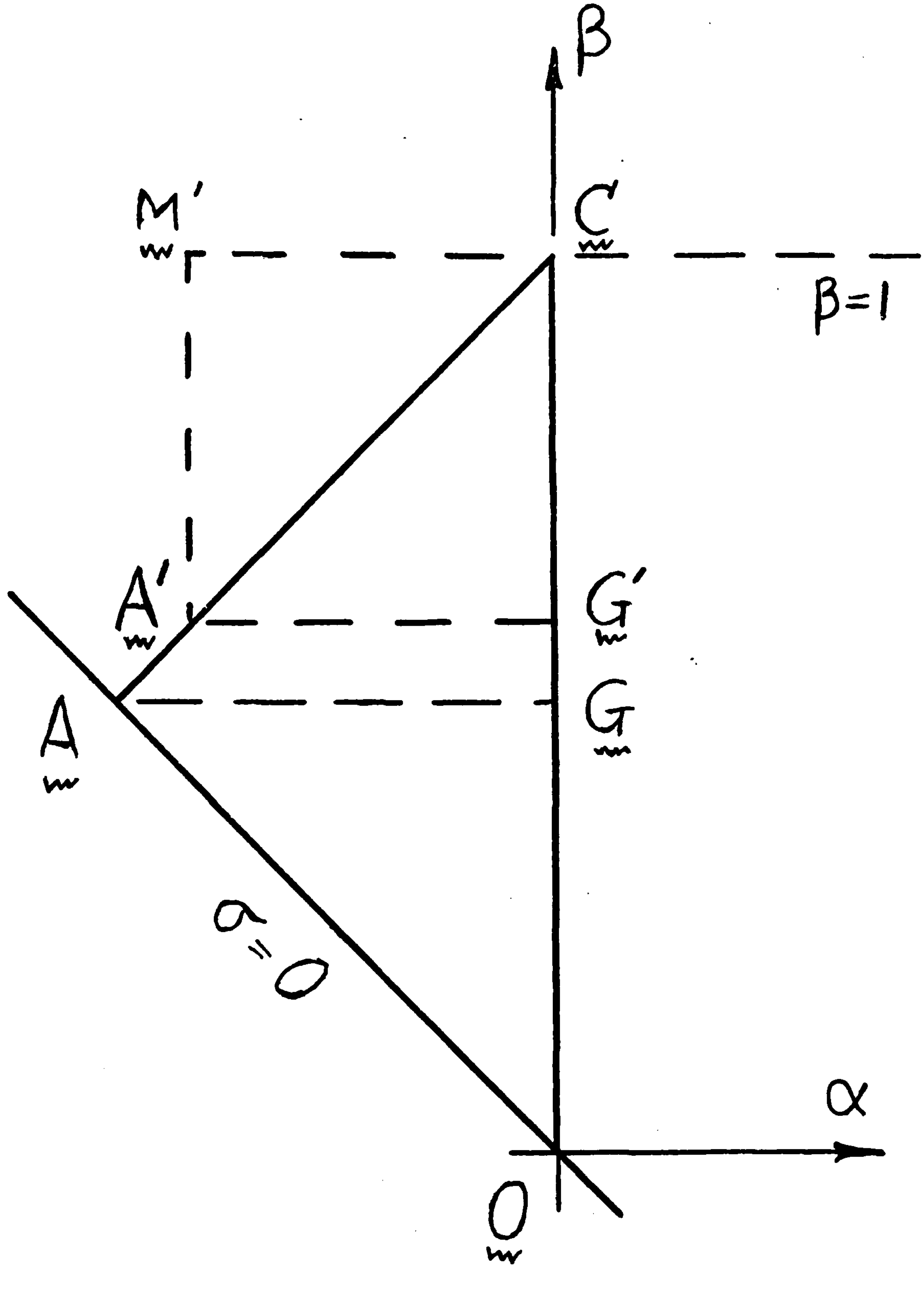

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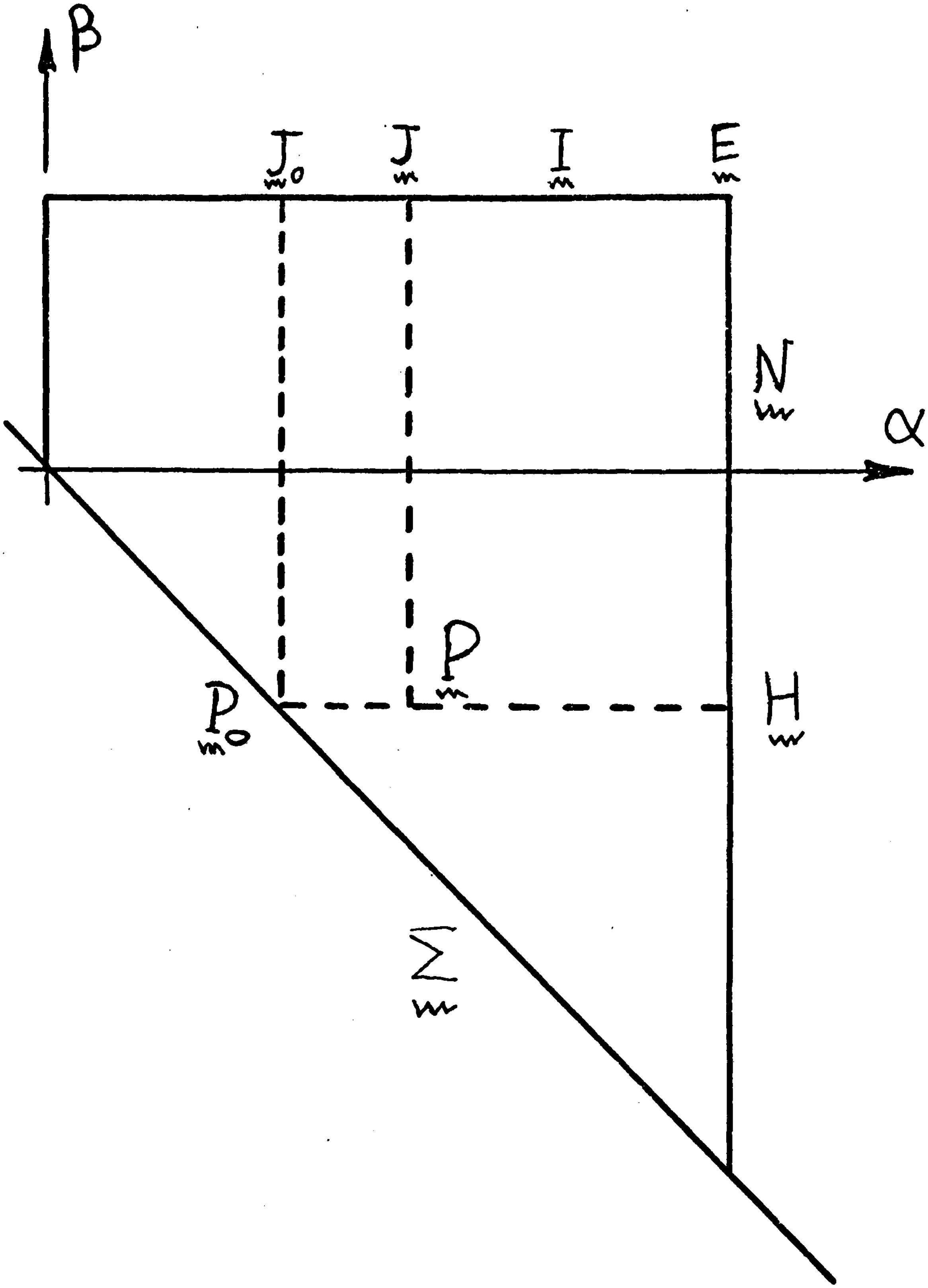

Figure 8

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Nonlinear Partial Differential Equations
Waves on Beaches

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The main structure underlying the nonlinearity of conservation laws of gasdynamical type in two independent variables is discussed at the hand of a canonical example describing also properties of water waves near shore. The ultimately singular nature of such laws is here the central issue and calls for an unusual formulation. Attention is directed to the globally strong

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20. ABSTRACT - cont'd.

solutions, and an unusual regularization is employed to make them accessible, after illposedness is overcome. The usual regularity theory is not normally sufficient for singular partial differential equations, and the necessary additional chapter on extensions to the singular locus is developed in detail for the canonical example. Criteria for the relation between regularized and strong solutions are discussed and used to characterize the class of solutions that are globally strong in the strictest sense.

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